A Moment Expansion of Downside Risk Measures

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Abstract
A decomposition of a sub-class of spectral risk measures is introduced in terms of L-moments accounting for geometric characteristics of the return distribution similar to the ones described by the ordinary moments. The decomposition completely characterises the spectral risk measures with square-integrable risk aversion functions and can be regarded as a link between higher-order moment risk and downside risk measures. Coherent approximations based on only a few L-moments can be successfully constructed for continuous risk aversion functions and can be applied to problems in portfolio theory to analyse the incremental impact of higher order moments on optimal allocations.

Keywords spectral risk measures, L-moments, coherent risk measures, mean-variance
1. Introduction
Although attempts to measure risk started as early as 1950s with the seminal paper by Markowitz (1952), who proposed variance as a proxy for risk, a more systematic approach was not undertaken until the introduction of coherent risk measures by Artzner et al. (1998). Even though different alternatives to variance had been considered, Artzner et al. (1998) put the notion of risk measure on an axiomatic basis introducing the concept of the axioms of coherency. As a particular representative, Conditional Value-at-Risk (CVaR) was suggested as a coherent alternative to Value-at-Risk (VaR) since VaR violates one of the coherency properties known as the sub-additivity property. CVaR allows for easy interpretation — it measures the average loss, provided that the loss is smaller than the VaR at a given tail probability. Apart from the intuitive definition, CVaR allows for linear relaxations in the context of optimal portfolio construction, (see Rockafellar and Uryasev, 2002). The concept of CVaR was generalised to spectral risk measures by Acerbi (2002). Instead of average losses beyond VaR, Acerbi suggested a weighted average of losses in which the weights satisfy certain properties, in order for the risk measure to be coherent.

Apart from these developments in academic literature, VaR and variance continue to be widely used in the industry. Spectral risk measures, and even CVaR, have not been adopted on a large scale. One reason is that it is difficult to relate them to familiar concepts such as any links with the moments of the probability distribution which capture intuitive geometric characteristics.

There have been efforts in this direction with respect to VaR and CVaR. Examples include modified VaR (mVaR) and modified CVaR (mCVaR) which are based on the Cornish-Fisher expansion of the quantile function and are representable in terms of the moments of the return distribution. The expression for mVaR includes the first four moments,

\[
mVaR(\epsilon) = \mu - \left( z_\epsilon + \frac{1}{6}(z_\epsilon^2 - 1)s + \frac{1}{24}(z_\epsilon^3 - 3z_\epsilon)s + \frac{1}{36}(2z_\epsilon^3 - 5z_\epsilon)s^2 \right) \sigma
\]

where \( z_\epsilon \) is the 1— \( \epsilon \)-quantile of the standard normal distribution, \( s \) is the skewness of \( X \), and \( k \) is the excess kurtosis of \( X \). A similar expression is derived for mCVaR in Boudt et al. (2008). mVaR can be regarded as a Gaussian VaR with a few correction terms to account for the presence of skewness and kurtosis in the data.

The advantage of those expressions is that they represent a simple relationship with concepts that are already well understood by practitioners. The drawback is that important properties are generally lost because the formulae are based on an asymptotic expansion. Thus, even though both mVaR and mCVaR can be viewed as approximations to VaR and CVaR, they do not have the appealing properties of the quantities they approximate. Moreover, the Cornish-Fisher expansion is sufficiently good for moderate departures from normality and for values of \( \epsilon \) which are not too small. As a result, if the tails of the return distribution are relatively fatter and/or if we want to approximate the VaR or CVaR at a small probability level, the mVaR and mCVaR approach may not be reliable.

Clearly, arriving at a simpler and more intuitive expression for a risk measure is of practical significance. In this paper, we consider a decomposition of a sub-class of spectral risk measures based on L-moments. The sub-class includes spectral risk measures with square-integrable weighting functions which, basically, includes all interesting cases from a practical viewpoint. The decomposition itself is a linear combination of L-moments in which the coefficients arise from the Legendre expansion of the weighting function which appears in the definition of spectral risk measures.

L-moments were first introduced by Sillito (1951) and later popularised by Hosking (1990). They describe geometric properties of the probability distribution of a random variable just like the
ordinary moments. They are theoretically more appealing due to their relative robustness and, just like the ordinary moments, they are easy to interpret. Further on, any random variable with a finite mean can be characterised in terms of its L-moments and, therefore, they represent another way of describing the distributional characteristics without any loss of information. In the field of finance, they are applied in the context of portfolio selection in Maillet and Merlin (2009) using multi-objective programming. Suggestions for applications in the field of finance are also available in Serfling and Xiao (2007).

L-moment expansions represent a new way of looking at spectral risk measures which is appealing from both a theoretical and a practical viewpoint. From a theoretical viewpoint, L-moment decompositions allow for a weak relation between spectral risk measures and utility functions. In fact, the construct of spectral risk measures reminds of a utility function. Dowd et al. (2008) argue that spectral risk measures can accommodate for investors with different levels of risk aversion and, potentially, the risk aversion level can be inferred from a utility function. Even though there are difficulties in establishing a functional connection, the L-moment decomposition implies that both quantities can be represented in terms of moment expansions, based on a Taylor series expansion and a Legendre series expansion, respectively. This implies that the way a spectral risk measure incorporates the geometric characteristics of the return distribution is similar to the way they are incorporated in expected utility leading to a weak link between the two concepts.

From a pure risk measurement perspective, the L-moment expansion indicates that a functional link between higher-order moment risk and downside risk measures can be established. On an intuitive level, a link of this type is not surprising and a consistency with higher-order moments is expected for any reasonable downside risk measure in the sense that higher skewness and/or smaller kurtosis should imply smaller risk. It is, however, surprising that measuring appropriately higher-order moments can result in a coherent risk measure. Further on, the framework allows to analyse which spectral risk measures result in non-intuitive sensitivities with respect to higher-order moments (e.g. a positive rather than a negative sensitivity to skewness).

In the context of portfolio construction, an L-moment expansion allows for an incremental analysis of the optimal portfolio composition in a general mean-risk framework by including terms from the expansion in an incremental manner. In this way, starting from the mean–Gini analysis, which is the second-order approximation and is the counterpart of mean–variance analysis in the expected utility framework, we can proceed to the third-order approximation and compare the differences in the optimal solutions. Likewise, we can compare the incremental change in the optimal solution of the fourth-order approximation relative to the third-order one and so on. Since the optimal solution of the $n$-th order approximation eventually converges to the corresponding optimal allocation of the spectral measure of risk, this analysis can help identify the opportunity cost of (not) considering the effect of higher order moments. Certainly, the opportunity cost depends on two factors — (i) the deviation from normality of the historical data and (ii) the sensitivity of the investor with respect to the higher-order moments.

The paper is organised in the following way. In the next section, we provide a brief description of spectral risk measures and L-moments. We proceed with the L-moment decomposition and consider three special cases – exponential and power spectral risk measures, and CVaR. Next, we provide a direct construction of coherent L-moment decompositions based on the first four L-moments describing mean, scale, skewness and kurtosis. Finally, we consider two implications for the coherency properties when building a risk measure directly from higher-order moments and comment on using the L-moment expansion in portfolio construction.

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1 - Characterising a probability distribution in terms of its moments is a well-known problem in probability theory. Using the ordinary moments, however, can result in an indeterminate moment problem, see Akhiezer (1965).
2. Spectral Risk Measures
Spectral risk measures were introduced by Acerbi (2002), extending the concept of CVaR. Instead of considering the average losses beyond VaR, Acerbi looked at family of risk measures based on a weighted average of quantiles,

\[ R_\phi(X) = -\int_0^1 F_{X}^{-1}(u) \phi(u) du, \]

in which the weighting function \( \phi(u) \) should satisfy certain properties in order for the weighted average to be a coherent risk measure. If these properties are satisfied, then \( \phi(u) \) is called a risk spectrum, and \( \rho_\phi \) is called a spectral risk measure. In this setting, CVaR is a spectral risk measure with a constant risk aversion function.

The properties that \( \phi(u) \) needs to satisfy are provided below.
- **Positivity:** \( \int_I \phi(u) du \geq 0 \), if \( I \subset [0, 1] \).
- **Non-increasing property:** \( \int_{q-\epsilon}^{q} \phi(u) du \geq \int_{q}^{q+\epsilon} \phi(u) \) for any \( q \in (0, 1) \) and any \( \epsilon > 0 \) such that \( [q - \epsilon, q + \epsilon] \subset [0, 1] \).
- **Normed:** all weights should sum up to 1, \( \int_0^1 \phi(u) du = 1 \).

The first and last properties indicate that \( \phi(u) \) is a weighting function — it is non-negative and sums up to one. The second property implies that larger losses appear with higher weights in the weighted average, which is a way of expressing risk aversion of investors. For this reason, we call \( \phi(u) \) a risk aversion function in this paper. Finally, the last property indicates that \( \phi(u) \) is generally assumed to be an element of the space \( L^1(0; 1) \).

Acerbi (2002) demonstrated that not only are these properties sufficient for \( \rho_\phi \) to be a coherent risk measure, but they are also necessary. Thus, any functional of the form of \( \rho_\phi \) is a coherent risk measure if and only if the weighting function satisfies these properties. There are a few families of spectral risk measures considered in the academic literature, such as CVaR, the exponential, and the power spectral risk measures, see Dowd et al. (2008).

3. L-moments
In this section, we review L-moments as they are instrumental in the L-moment decomposition. Generally, the ordinary centered moments of a random variable are characteristics describing certain geometric properties of the probability distribution. Thus, standard deviation describes scale, the third centered moment can be used to describe skewness and the fourth centered moment can describe kurtosis. In a similar way, L-moments represent another set of characteristics which describe the same geometric properties. L-moments are defined as linear combinations of order statistics,

\[ \lambda_1 = E(X_{1:1}) = EX, \]
\[ \lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}), \]
\[ \lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3}), \]

and in general

\[ \lambda_r = \frac{1}{r} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} E(X_{r-j:r}). \]

where \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) represent order statistics of a one-dimensional random variable \( X \).
Equivalently, they can be introduced as weighted averages of the quantile function in which the weights are determined by the shifted Legendre polynomials,

$$\lambda_r = \int_0^1 F^{-1}(u) P_{r-1}^*(u) du$$

where $F^{-1}(u) = \inf\{x : P(X \leq x) \geq u\}$ is the inverse distribution function of $X$ and

$$P_r^*(u) = \sum_{k=0}^r \tilde{p}_{r,k} u^k$$

in which

$$\tilde{p}_{r,k} = (-1)^{r-k} \binom{r}{k} \binom{r+k}{k} = \frac{(-1)^{r-k}(r+k)!}{(k!)^2(r-k)!}.$$ 

For a detailed description of L-moments, see Hosking (2005).

Some properties of the shifted Legendre polynomials which will be used in the discussion are summarised below:

1. $P_r^*(u)$... forms a complete, orthogonal system in $L^2(0, 1)$.

More precisely,

$$\int_0^1 P_m^*(u) P_n^*(u) du = \frac{1}{2n+1} \delta_{mn},$$

where $\delta_{mn}$ denotes the Kroneker delta. Due to the completeness property, any square integrable function $f \in L^2(0, 1)$ can be expressed as a linear combination of Legendre polynomials through the so-called Legendre expansion.

2. The shifted Legendre polynomials are scaled, so that $P_r^*(1) = 1, r = 0, 2, \ldots$ and they are symmetric or anti-symmetric, $P_r^*(1-2u) = (-1)^r P_r^*(2u-1)$, which implies that $P_r^*(0) = (-1)^r$.

3. The first four shifted Legendre polynomials are

$$P_0^*(u) = 1$$
$$P_1^*(u) = 2u - 1$$
$$P_2^*(u) = 6u^2 - 6u + 1$$
$$P_3^*(u) = 20u^3 - 30u^2 + 12u - 1$$

As a consequence of the orthogonality property, all polynomials with the exception of $P_0^*(u)$ integrate to zero,

$$\int_0^1 P_n^*(u) du = 0, \quad n = 1, 2, \ldots$$

The important properties of L-moments can be summarised as follows.

1. If $E|X| < \infty$, then $\lambda_r < \infty$.
2. If $E|X| < \infty$, then $\lambda_r, r = 1, \ldots$ uniquely define the probability distribution in the sense that two distinct distributions cannot have the same L-moments.
3. If $X$ is a symmetric random variable with a finite mean, then all odd L-moments are equal to zero, $\lambda_r = 0, r = 3, 5, \ldots$.
4. Concerning the shape of the probability distribution, $\lambda_2$ describes scale, $\lambda_3$ can be used to describe skewness, and $\lambda_4$ can describe kurtosis.
5. $\lambda_2$ is a scalar multiple of the expectation of Gini’s mean difference statistic. In particular, the second sample L-moment equals half of the Gini measure, $\lambda_2 = \frac{1}{2} G$, where

$$G = \left( \frac{n}{2} \right)^{-1} \sum_{1 \leq i < j \leq n} (x_{j:n} - x_{i:n})$$
Proofs, estimators and further properties can be found in Hosking (1990) and Hosking (2005). Multivariate extensions can be found in Serfling and Xiao (2007) and applications in finance are available in Serfling and Xiao (2007) and Maillet and Merlin (2009).

4. L-moment Expansions of Spectral Risk Measures

Any risk measure incorporates two important aspects — (1) the characteristics of the return distribution and (2) the subjective attitude of the investor towards these characteristics. As far as the first aspect is concerned, it is captured by the quantile function in the case of spectral risk measures which provides a full description of the distributional characteristics.

The second aspect is captured in different ways. In the case of VaR, for example, it is taken into account through the confidence level parameter indicating which quantile is important and the remaining quantiles are not considered at all. In the more sophisticated case of spectral risk measures, it is described by the function \( \phi(u) \) which indicates how a given investor weighs the quantiles. The higher the risk aversion of an investor, the larger weights would be assigned to the lower quantiles of the return distribution, which represent higher losses.

An L-moment expansion allows for an alternative interpretation. The first aspect is captured by the L-moments of the return distribution. Since any random variable is uniquely described in terms of its L-moments, the description of the distributional properties is complete. The second aspect is captured by the coefficients in the expansion which represent sensitivities with respect to the geometric characteristics captured by the L-moments. Since they completely describe the function \( \phi(u) \) through the Legendre series expansion, there is no loss of information and the two ways of calculating the risk measure are equivalent.

In this section, first we introduce a functional which we call an L-moment representation of a spectral risk measure. Next, we demonstrate that it describes the set of spectral risk measures with square-integrable risk aversion functions which is a consequence of the completeness of shifted Legendre polynomials. In all results in this section, we assume that \( \phi \in L^2(0, 1) \subset L^1(0, 1) \), which is a sub-set of all possible risk aversion functions.

**Definition 4.1.** The functional \( \mathcal{R}_\lambda(X) \),

\[
\mathcal{R}_\lambda(X) = D_\lambda(X) - EX
\]

where \( D_\lambda(X) = \sum_{k=2}^{\infty} g_k \lambda_k \), in which \( \lambda_k \) are the L-moments of \( X \) defined in (3.1) is called an L-moment decomposition, or an L-moment expansion, where the functional \( D_\lambda(X) \) can be equivalently represented as

\[
D_\lambda(X) = \int_0^1 F_X^{-1}(u) g_{\lambda}(u) du,
\]

in which \( g_{\lambda}(u) = \sum_{k=2}^{\infty} a_k P_{k-1}^*(u) \) where \( P_{k-1}^*(u) \) is a shifted Legendre polynomial.

The following representation result holds.

**Proposition 4.1.** Let \( \mathcal{R}_\phi(X) \) be a spectral risk measure,

\[
\mathcal{R}_\phi(X) = -\int_0^1 F_X^{-1}(u) \phi(u) du,
\]

where \( \phi(u) \) is the risk aversion function. If \( \phi \in L^2(0, 1) \), then \( \mathcal{R}_\phi(X) \) allows for a L-moment representation in which

\[
a_k = -(2k - 1) \int_0^1 \phi(u) P_{k-1}^*(u) du.
\]
Conversely, any L-moment representation $R_\lambda(X)$ defined in (4.1) is also a spectral risk measure with

$$ (4.5) \quad \phi_\lambda(u) = 1 - \sum_{k=2}^{\infty} a_k P_{k-1}^*(u) $$

where $P_{k-1}^*(u)$ are the shifted Legendre polynomials provided that $\phi_\lambda(u)$ is a risk aversion function.

**Proof.** From the orthogonal property and the completeness of the shifted Legendre polynomials in $L^2(0, 1)$, it follows that

$$ \phi(u) = \sum_{i=0}^{\infty} b_i P_i^*(u) $$

where $b_i = (2i + 1)$

Since $P_0(u) = 1$, we obtain that $b_0 = \int_0^1 \phi(u)du = 1$ where the last equality is a property of the risk aversion function. As a consequence,

$$ \mathcal{R}_\phi(X) = -\int_0^1 F_X^{-1}(u) \left[ \sum_{i=0}^{\infty} b_i P_i^*(u) \right] du $$

$$ = \sum_{i=1}^{\infty} (-b_i) \int_0^1 F_X^{-1}(u) P_i^*(u) du - b_0 EX $$

$$ = \sum_{i=2}^{\infty} a_i \lambda_i - b_0 EX $$

where $a_i = -b_{i-1}$.

The converse statement is a simple consequence of the representation

$$ \mathcal{R}_\lambda(X) = -\int_0^1 F_X^{-1}(u)(1 - g_\lambda(u))du $$

where $g_\lambda(u) = \sum_{i=0}^{\infty} b_i P_i^*(u)$ which follows directly from the definitions in (3.1) and (4.1). Acerbi (2002) demonstrates that functionals of this type are coherent risk measures if and only if the weighting function $(1 - g_\lambda(u))$ satisfies the properties of risk aversion functions.

The coefficients $a_k$ can also be calculated in a different way if the risk aversion function $\phi(u)$ is sufficiently smooth.

**Proposition 4.2.** Suppose that $\phi(u) \in L^2(0, 1)$ and has a $k$-th derivative which is itself an integrable function. Then

$$ (4.6) \quad a_k = -\frac{2k - 1}{(k-1)!} \int_0^1 \phi^{(k-1)}(u)(u - u^2)^{k-1} du $$

Further on, a simple sufficient condition on the signs of the coefficients is the following one: $a_k$ is positive if $\phi^{(k-1)}(u) \leq 0, \forall u \in [0, 1]$. The coefficient $a_k$ is bounded by

$$ (4.7) \quad |a_k| \leq \frac{2k - 1}{4^{k-1}(k-1)!} \int_0^1 |\phi^{(k-1)}(u)|du. $$
Proof. The proof is a simple consequence of the Rodriguez representation of the shifted Legendre polynomials,

$$P^*_k(u) = \frac{d^k}{du^k} (x - x^2)^k.$$  

Integration by parts proves the result. The sufficient condition on the sign of $a_k$ follows from the fact that the function $(u - u^2)^k$ is positive for any $u \in [0, 1]$. The bound follows after observing that the maximum of this function is attained at $u = 1/2$ and equals $1/4$.

Generally, there is agreement in recent empirical studies that investors prefer smaller even moment (e.g. variance, kurtosis, etc.) and larger odd moments (e.g. mean, skewness, etc.). The result in Proposition 4.2 implies that this preference relation is incorporated in the risk measure for investors having a sufficiently smooth risk aversion function with an alternating derivative sign.

Further on, the bound in (4.7) can be used to assess the approximate value of the sensitivity to the corresponding L-moment in the expansion. Further on, if the derivatives are themselves bounded, it shows a fast rate of decay of the corresponding sensitivity meaning that the coefficients rapidly decrease because of the explosion of the denominator.

The completeness of the shifted Legendre polynomials in $L^2(0, 1)$ implies that spectral risk measures can be approximated by a sequence of L-moments in terms of the $L_2$ norm,

$$\lim_{j \to \infty} \|\phi_j - \phi\|_2 = 0$$  

where $\phi_j(u) = \sum_{i=0}^{j} a_i P^*_i(u)$ and $a_k$ are the calculated as in (4.4). The $L_2$ norm convergence emphasises that the approximation of the risk aversion function is "on average". Approximating the risk aversion function generally makes sense because of the following more general result.

Proposition 4.3. Let $\mathcal{R}_\phi(X)$ be a spectral risk measure of an random variable $X$. If we have a sequence of risk spectrum functions $\{\phi_k\}^\infty_{k=1}$, $\phi_i \in L^k(0, 1)$ and $\|\phi_j - \phi\|_k < \epsilon$, for all $j > N_\epsilon \in \mathbb{N}$, where $\phi \in L^k(0, 1)$ is a risk spectrum, then

$$|\mathcal{R}_{\phi_j}(X) - \mathcal{R}_\phi(X)| < C\epsilon, \quad \text{for all } j > N_\epsilon \in \mathbb{N}$$

where $C = (E|X|)^{1/s}$, $s = k/(k - 1)$

Proof.

$$|\mathcal{R}_{\phi_j}(X) - \mathcal{R}_\phi(X)| = \left| \int_0^1 F_X^{-1}(u)(\phi(u) - \phi_j(u)) \, du \right|$$  

$$\leq \int_0^1 |F_X^{-1}(u)||\phi(u) - \phi_j(u)| \, du$$  

$$\leq \left( \int_0^1 |F_X^{-1}(u)|^s \, du \right)^{1/s} \left( \int_0^1 |\phi(u) - \phi_j(u)|^k \, du \right)^{1/k}$$  

$$= (E|X|)^{1/s} ||\phi(u) - \phi_j(u)||_k$$

where $\frac{1}{s} + \frac{1}{k} = 1$ and the second inequality follows from Hölder’s inequality. Since by assumption the risk spectrum $\phi$ is the limit in $||\cdot||_k$ sense, for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$, such that $||\phi_j - \phi||_k < \epsilon$ for all $j > N_\epsilon$ and we receive the desired bound.

In the context of L-moment expansions, this result applies with $k = 2$, which means that reliable approximations can be constructed for return distributions with a finite volatility. For bounded risk aversion functions, however, the result applies with $s = 1$, i.e. if $\phi \in L^\infty(0, 1)$. Therefore, if the risk aversion function is bounded, a finite expectation of the return distribution is the condition that guarantees a reliable approximation.

2 - For additional information, see the discussion in Kleniati and Rustem (2009).
Due to the fact that the approximation of the risk aversion function is obtained through the $L_2$ norm and works "on average", a partial L-moment expansion based on the approximation of the first $n$ L-moments,
\[
R^n(X) = \sum_{k=2}^{n} a_k \lambda_k - EX
\]
may not be a coherent risk measure. The $n$-th order approximation $\phi^n(u) = 1 - \sum_{k=2}^{n} a_k P^*_k(u)$ may violate some of the properties of risk aversion functions. The conditions that guarantee that the $n$-th order approximation is a coherent risk measure are rather difficult to translate into conditions only in terms of the coefficients of the expansion. We continue this discussion in Section 6 by providing an explicit construction in terms of the first four L-moments.

Generally, the assumption that $\phi \in L^2(0, 1)$ means that the L-moment expansion works for a sub-class of spectral risk measures. This assumption can be interpreted in the following way in terms of the L-moment sensitivities. The squared norm $||\phi||_2^2$ which is assumed to be finite has a representation in terms of the coefficients $a_k$ and can be bounded by the total squared sensitivity,
\[
||\phi||_2^2 = 1 + \sum_{k=1}^{\infty} \left( \frac{a_k}{2k-1} \right)^2 \leq 1 + \sum_{k=1}^{\infty} a_k^2.
\]
Therefore, the assumption $||\phi||_2 < \infty$ is guaranteed by a finite total squared sensitivity which implies that the squared individual sensitivities decay, i.e. the risk measure is relatively less sensitive to the higher order L-moments. If $\phi \notin L^2(0, 1)$ and we attempt to construct the L-moment decomposition, then the sensitivities will still be finite for any finite $k$ because
\[
|a_k| \leq (2k-1) \int_0^1 \phi(u) |P^*_k(u)| du \leq 2k - 1,
\]
the total squared sensitivity, however, explodes implying that the individual sensitivities do not decay fast enough.

In the next section, we discuss particular spectral risk measures and their L-moment decomposition. We consider an example allowing for a straightforward decomposition based on an exponential risk-aversion function and the case of CVaR where the decomposition fails due to the so-called Gibbs phenomenon which is widely studied in the area of signal processing.

5. Examples of L-moment Decompositions
In this section, we consider three examples for risk aversion functions and the moment decompositions of the corresponding spectral risk measures — an exponential, a power and CVaR which is an example of a step risk aversion function. An L-moment decomposition provides a way of determining the risk aversion parameters in the exponential and the power families by linking it to a desired level of sensitivity to a particular geometric characteristic as described by the corresponding L-moment. Even though we consider two special cases, the theory is applicable with any other parametric family of sufficiently smooth functions.

The fact that smoothness is important is demonstrated by the CVaR example. Discontinuities in the risk aversion function lead to a pathology known as the Gibbs phenomenon in signal processing.

5.1 Exponential risk aversion
An example of a family of risk aversion functions considered in Dowd et al. (2008) and Scherer (2009) is based on an exponential risk aversion function,
where $\gamma > 0$. The parameter $\gamma$ is the coefficient of risk aversion.

The corresponding coefficients in the L-moment decomposition are calculated in the following proposition.

**Proposition 5.1.** The coefficients in (4.4) of a spectral risk measure with a risk aversion function given in (5.1) equal

\[
(5.2) \quad a_{k,\gamma} = - (2k - 1)p^*_{k-1,0}
\]

\[
\quad + (2k - 1) \sum_{i=1}^{k-1} p^*_{k-1,i} \frac{\gamma}{e^{\gamma} - 1} \left[ \sum_{j=1}^{i} \frac{i!}{(i-j+1)!j!} - \frac{i!}{\gamma^i} I_0 \right],
\]

where $I_0 = \frac{1}{\gamma}(e^{\gamma} - 1)$ and $p^*_{k-1,i}$ are the coefficients of the Legendre polynomials given in (3.3).

**Proof.** The result follows directly from the definitions in (4.4) and (3.2),

\[
\begin{align*}
\quad a_k &= - (2k - 1) \int_0^1 \phi(u) P^*_{k-1}(u) du \\
\quad &= - (2k - 1) \sum_{i=0}^{k-1} p^*_{k-1,i} \int_0^1 \phi(u) u^i du \\
\quad &= - (2k - 1) \frac{\gamma}{e^{\gamma} - 1} \sum_{i=0}^{k-1} p^*_{k-1,i} \int_0^1 e^{\gamma(1-u)} u^i du.
\end{align*}
\]

The expression in (5.2) is derived by computing the integrals $\int_0^1 e^{\gamma(1-u)} u^i du$ by integration by parts.

Figure 1 shows $\phi_{\exp}(u, \gamma)$ together with the sensitivities of the first 20 L-moments in the case of $\gamma = 2$ and $\gamma = 8$. Increasing the value of the risk aversion parameter results in a steeper risk aversion function. In the case of $\gamma = 2$, the coefficient of the scale parameter as computed by $\lambda_2$ is the largest one in absolute value meaning that for this risk measure, the scale parameter seems to be the most important geometric characteristic. Increasing the value to 8, however, leads to a higher relative sensitivity with respect to skewness. Therefore, increasing the value of the risk aversion parameter puts more emphasis on the higher order moments.

The coefficient $a_{k,\gamma}$ can also be viewed as a function of the risk aversion parameter $\gamma$ (see Figure 2). Therefore, using the expression in (5.2), it is possible to calibrate the value of the risk aversion by the magnitude of the desired sensitivity with respect to a given geometric characteristic. For example, if a high sensitivity with respect to the fifth L-moment is not desirable, i.e. we would like to limit the expansion up to the first four moments, then we can solve $a_{k,\gamma} = \varepsilon$ for $\gamma$ and verify additionally that the partial L-moment decomposition based on the first four moments is coherent.

5.2 Power risk aversion functions

Power risk aversion functions are defined as,

\[
\phi_{\text{pow}}(u) = \alpha(1-u)^{\alpha-1}, \quad u \in [0,1]
\]

where $\alpha \geq 1$ is the relative risk aversion parameter, see Dowd et al. (2008).
Figure 1: Two examples of exponential risk aversion functions with the corresponding L-moment sensitivities.

Figure 2: The coefficients $a_{k,\gamma}$ of the exponential risk aversion function as a function of $\gamma$. 
Figure 3: Two examples of power risk aversion functions with the corresponding L-moment sensitivities.

![Graph of power risk aversion functions and L-moment sensitivities]

The analogue to Proposition 5.1 is provided below.

**Proposition 5.2.** The coefficients in (4.4) of a spectral risk measure with a power risk aversion function equal

\[
(5.3) \quad a_{k,\alpha} = -(2k-1) \sum_{i=0}^{k-1} p_{k-1,i}^2 \frac{\Gamma(i+1)\Gamma(\alpha)}{\Gamma(\alpha+i+1)},
\]

where \( p_{k-1,i} \) are the coefficients of the Legendre polynomials given in (3.3).

**Proof.** Using the same steps as in the proof of Proposition 5.1, we arrive at the expression

\[
a_k = -(2k-1) \sum_{i=0}^{k-1} p_{k-1,i}^2 \int_0^1 \alpha(1-u)^{\alpha-1}u^i du,
\]

in which the integral \( \int_0^1 (1-u)^{\alpha-1}u^i du = \frac{\Gamma(i+1)\Gamma(\alpha)}{\Gamma(\alpha+i+1)} \) defines the beta function.

Figure 3 shows two examples of power risk aversion functions and the corresponding L-moment sensitivities. Power risk aversion functions allow for both concave (if \( 1 \leq \alpha \leq 2 \)) and convex representatives (if \( \alpha \geq 2 \)). The bottom plot in Figure 3 shows the change of sign of the sensitivity to the third L-moment when the shape of the risk aversion function changes from concave to convex. Like the exponential risk aversion function, the higher the curvature of the function (the higher the second derivative), the higher the contribution of the third L-moment in the expansion.

The sensitivities to the first few L-moments of the power risk aversion function regarded as a function of the parameter \( \alpha \) are provided in Figure 4. The functions are not monotonic because of the change of the shape of \( \phi_{\text{pow}}(u, \alpha) \) from concave to convex. If \( \alpha = 2 \), then \( \phi_{\text{pow}}(u, \alpha) \) is linear.
and the sensitivities to $\lambda_3$ and above are equal to zero. If $\alpha = 3$, then $\phi_{\text{pow}}(u, \alpha)$ is quadratic and the sensitivities of $\lambda_4$ and above are equal to zero. This behavior appears because the power function can be expressed as a polynomial when $\alpha$ is an integer and for this reason the coefficients of the polynomials in the Legendre expansion which include higher powers than $\alpha$ are equal to zero.

5.3 CVaR

Conditional value-at-risk (CVaR) is defined as the average VaR above a certain VaR level,

$$CVaR_\varepsilon(X) = -\frac{1}{\varepsilon} \int_0^\varepsilon F^{-1}_X(u)du,$$

where $F^{-1}_X(u)$ denotes the inverse of the distribution function of $X$. The parameter $0 < \varepsilon < 1$ is the tail probability parameter which is supposed to be a small number.

CVaR at tail probability $\varepsilon > 0$ is a spectral risk measure with a risk aversion function $\phi(u) = \frac{1}{\varepsilon} I_{[0, \varepsilon]}$ where $I_{[0, \varepsilon]}$ denotes the indicator function of the interval $[0, \varepsilon]$. Orthogonal polynomial approximations to step functions and, generally to functions with discontinuities, fail with a pathology known as the Gibbs phenomenon. It materialises as an oscillation close to the point of discontinuity decaying away from it.

The Gibbs phenomenon is illustrated on Figure 5. The figure demonstrates that a decomposition based on the first few L-moments is not a coherent risk measure even though at the limit the L-moment approximation becomes coherent. The properties that fail are the sub-additivity property, because the polynomial approximation is non-monotonic, and the monotonicity property, because the polynomial approximation may become negative.

Figure 4: The coefficients $a_{k, \alpha}$ as a function of $\alpha$ of the exponential risk aversion function.

Figure 5: The risk aversion function of CVaR at 25% tail probability together with 10 (u), 15(u), and 20 (u) including the first $\phi_{\lambda}^{10}(u)$, $\phi_{\lambda}^{15}(u)$, and $\phi_{\lambda}^{20}(u)$ L-moments respectively.
6. Direct Construction of Coherent L-Moment Decompositions

As noted in Section 4, direct construction of an L-moment decomposition of a spectral risk measure is difficult because the coherency properties are difficult to state in terms of the L-moment sensitivities alone. However, it turns out that a direct construction is feasible on the basis of the first four moments — mean, scale, skewness and kurtosis. In this section, we begin with a more general result and then move on to the more simple case. We provide an intuitive explanation of how the simpler case works.

Proposition 6.1. The functional $\mathcal{R}_\lambda(X)$ defined in (4.1) is a spectral risk measure if $\sum_{k=2}^{\infty} a_k \in [0, 1] / and \sum_{k=2}^{\infty} a_k dP_{k-1}(u)/du \geq 0$ where $P_{k-1}(u)$ are the Legendre polynomials defined in (3.2).

Proof. The differential condition guarantees a non-increasing risk aversion function of the L-moment decomposition given in (4.5). The other condition follows from the requirement for non-negativity of the risk aversion function and the property $P_k(1) = 1, k = 0, 1, \ldots$ of the shifted Legendre polynomials.

The first condition in the proposition is easy to check. The second one, however, is a complicated expression involving the derivatives of the shifted Legendre polynomials and it is difficult to reformulate a sufficient condition involving the sensitivities $a_k$ only. Limiting the discussion to the second-order approximation only, we obtain the following interesting corollary.

Corollary 6.1. The second-order approximation to any spectral risk measure with a square-integrable risk aversion function is given by $\mathcal{R}_\lambda^2(X) = a_2 - E X$. It is coherent if $a \in [0, 1]$.

Since $\lambda_2$ is connected to the Gini coefficient of dissimilarity, the corollary can be another motivation for the application of the Gini coefficient in finance.

If we limit ourselves to the first four L-moments representing measures of mean, scale, skewness and kurtosis respectively, we can arrive at simpler conditions than the ones included in Proposition 6.1.

Proposition 6.2. Let $a_2 = 1/(1 + \beta + \gamma), \ a_3 = -a_2\beta, \ a_4 = a_2\gamma, \ where \ \beta \in [0, 1/3] \ and \ \gamma \in [0, (5 + 2\sqrt{5})/15], \ and \ a_k = 0, k = 5, \ldots$ Then, $\mathcal{R}_\lambda(X)$ defined in (4.1) is a spectral risk measure. Further on, $\mathcal{R}_\lambda(X)$ is bounded if $E|X| < \infty$.

Proof. First, if $\beta = 0$ and $\gamma = 0$, then $\mathcal{R}_\lambda(X)$ is a spectral risk measure because the risk aversion function in (4.5) equals

$$\phi_\lambda(u) = 1 - a_2 P_1(u) = -2a_2 u + 1 + a_2,$$

and $a_2 \in [0, 1]$ from the conditions in the proposition.

The remaining cases are easier to prove by restating the coherency properties for $g_\lambda(u)$ defined in (4.1) rather than working with the corresponding risk aversion function $\phi_\lambda(u) = 1 - g_\lambda(u)$. The weighting function $g_\lambda(u)$ should be (1) non-decreasing and (2) should integrate to zero. Condition (2), however, is always satisfied due to the orthogonality of the polynomials and the fact that $P_0^*(u) = 1$.

Consider the case $\beta \neq 0$ and $\gamma = 0$. The weighting function $g_\lambda(u)$ has the form,

$$g_\lambda(u) = a_2(2u - 1) - a_3(6u^2 - 6u + 1).$$

The first derivative is a linear function and the condition $a_3 \leq a_2/3$ guarantees non-negativity of $g_\lambda'(u)$ for any $u \in [0, 1]$. 
Consider the general case $\beta \neq 0$ and $\gamma = 0$. The weighting function and its first derivative equal,

$$g_\lambda(u) = a_2(2u - 1) - a_3(6u^2 - 6u + 1) + a_4(20u^3 - 30u^2 + 12u - 1)$$

$$g'_\lambda(u) = 60a_4u^2 - 12u(a_3 + 5a_4) + 2a_2 + 6a_3 + 12a_4.$$ 

The first derivative is a quadratic function and since by assumption $a_4 \geq 0$, the quadratic function is convex. A non-positive discriminant guarantees non-negativity for any $u \in [0, 1]$. Representing $\delta = \delta a_3$, $\delta > 0$, using the inequality $a_3 \leq a_2/3$ from the case above and imposing a non-positive discriminant, we obtain $a_2 / a_3 \leq 10/(3\delta^2 + 15)$ where $\delta \in [0, 5 - 2\sqrt{5}]$. As a consequence, $g_\lambda(u)$ is increasing if $a_3 \leq a_2/3$ and $a_4 \leq a_2(5 + 2\sqrt{5})/15$.

If the $E[X] < \infty$ then all L-moments are finite. As a consequence, the risk measure is finite as well.

Figure 6: The weighting function $g_\lambda(u)$ for some extreme choices of the coefficients in Proposition 6.2.

An illustration of the function $g_\lambda(u)$ from the definition in (4.2) with the extreme values for the coefficients $a_3$ and $a_4$ from Proposition 6.2 is available in Figure 6. We plot $g_\lambda(u)$ instead of the risk spectrum $\phi_\lambda(u) = 1 - g_\lambda(u)$ because in this way we ignore the impact of the mean of $X$ and the plots are easier to interpret. The case with no skewness and kurtosis arises when $a_2 = 1$, $a_3 = 0$, and $a_4 = 0$. The function $g_\lambda(u)$ is linear which implies that the quantiles located symmetrically above and below the median get the same coefficients in absolute value.

The case with maximum sensitivity with respect to skewness and zero kurtosis sensitivity is obtained with $a_2 = 3/4$, $a_3 = 1/4$, and $a_4 = 0$. In this case, $g_\lambda(u)$ is quadratic with a maximum at $u = 1$. It is asymmetric in the sense that the left tail gets larger coefficients in absolute value than the right tail and therefore a positively skewed $X$ is less risky than a negatively skewed $X$.

Finally, the case with maximum kurtosis sensitivity and zero skewness sensitivity is obtained with $\gamma = (5 + 2\sqrt{5})/15$. The function $g_\lambda(u)$ is cubic and symmetric in absolute value around $u = 1/2$. It assigns coefficients close to zero for all quantiles around the median and heavily penalises the extreme quantiles. Therefore, a distribution which is less peaked around the mean and has less heavy tails is less risky even if we keep the scale, or $\lambda_2$, constant.

7. Comments on the Coherency Properties
The L-moment expansion of spectral risk measures provides a link between two seemingly unrelated aspects of risk — higher-order moment risk and downside risk. It illustrates that building a risk measure as a linear combination of higher order moments can result in a spectral risk measure which is a coherent downside risk measure on condition that the higher-order moments are
properly measured and if sensitivities are properly chosen. Therefore, these two notions of risk can coincide.

In this regard, there are two observations worth noting. First, using ordinary moments instead of L-moments does not generally lead to a downside risk measure in the sense of spectral risk measures. The application of ordinary moments is rationalised through an approximation of the expected utility function based on a Taylor series expansion,

$$EU(X) = \sum_{k=0}^{\infty} \frac{U^{(k)}(EX)}{k!} \mu_k,$$

where $$\mu_k = E(X - EX)^k$$ and $$U^{(k)}$$ denotes the $$k$$-th derivative of $$U(x)$$. Thus, the first few terms lead to an approximation of expected utility and it makes sense to consider the approximation as an objective in portfolio construction problems.5

Certainly, there is similarity between the Taylor series expansion of expected utility and the Legendre expansion of spectral risk measures,

$$\mathcal{R}_\phi(X) = -EX + \sum_{k=2}^{\infty} a_k \lambda_k,$$

where $$\lambda_k$$ is the $$k$$-th L-moment and $$a_k$$ is the coefficient given in (4.4). L-moments are, however, more robust than ordinary moments and represent another way of describing the geometric properties of the return distribution. Even though robustness is a desirable property, robustifying them further will lead to a violation of the sub-additivity property. That is, using the right expression $$\mathcal{R}_\phi(X)$$ but simply changing $$\lambda_k$$, for example, with a trimmed L-moment will lead to a violation. In some applications, the sub-additivity property has been disputed, but in the context of portfolio construction it is this property that ensures diversification opportunities would be recognized. Therefore, preserving this property for the purpose of portfolio construction makes good sense. For an additional discussion on the sub-additivity property, see Deguest et al. (2010).

More robust estimators can be used but then the coefficients need to be revised in order for the coherency properties to be preserved.

Second, using L-moments but assigning arbitrary sensitivities does not necessarily lead to a risk measure with good properties either — there could be a violation of the sub-additivity property (e.g. oscillating risk aversion function) and/or the monotonicity property (a negative risk aversion function). For example, looking at the simple case of mean, scale and skewness, increasing the skewness sensitivity too much can lead to a violation unless we include a higher-order L-moment in order to compensate for the violation.

8. Implications for Portfolio Construction

The classical mean-variance analysis arises as a second-order approximation to expected utility. In a similar manner, Corollary 6.1 demonstrates that the second-order approximation for spectral risk measures is the quantity

$$(8.1) \quad \mathcal{R}_\phi^2(X) = -EX + a \lambda_2, \quad a \in [0, 1]$$

By definition, $$\lambda_2 \geq 0$$ and, therefore, it is a deviation measure in the sense of Rockafellar et al. (2006). As a result, starting from $$n$$ risky assets, we can construct the mean-Gini efficient frontier by considering the problem

$$\begin{align*}
\min_{w} & \quad \lambda_2(w) \\
\text{s.t.} & \quad e' \mu = R
\end{align*}$$

where $$\mu$$ is the vector of means of the risky assets, $$\lambda_2(w)$$ denotes the second L-moment of the

---

5 - See Martellini and Ziemann (2010) for additional information.
6 - The mean-deviation efficient portfolios include the mean-risk efficient portfolios, see Rachev et al. (2008) for additional information. We call the efficient frontier mean-Gini because of the link between $$\lambda_2$$ and the Gini coefficient.
portfolio \( w, e = (1, \ldots, 1) \), and \( R \) is the target return. By varying \( R \), we obtain the entire set of efficient portfolios.

Note that (8.2) is one and the same for any spectral risk measure that allows for an L-moment expansion, implying that reducing scale is of high priority for risk minimisation. Therefore, if we are unsure about the choice for the risk aversion function in the definition of the risk measure, solving (8.2) would provide a proxy for a diversified portfolio. This optimisation problem is analogous to the celebrated mean-variance problem. The only difference is that instead of variance (or standard deviation for that purpose), we consider the second L-moment. However, just like standard deviation, \( \lambda_2 \) is a measure of scale, albeit a more robust one. In fact, the mean-Gini framework was developed by Shalit and Yitzhaki (1984) as an alternative to mean-variance analysis and our results indicate that it has the same central role as mean-variance for a big subclass of spectral risk measures.

In a similar fashion, we can add a skewness and a kurtosis component in the objective function in (8.2). Keeping the corresponding sensitivities within the bounds given in Proposition 6.1 guarantees that the optimisation problem is convex. The actual solution of the convex problem can be obtained either through the linear relaxation behind the spectral risk measure suggested by Acerbi and Simonetti (2002) or by solving directly the problem as formulated in terms of the L-moments. The objective function in (8.2) can be re-stated in terms of the co-L2-moments which are provided in Serfling and Xiao (2007).

Starting from the L-moment expansion of a given spectral risk measure, we can analyse the incremental impact of each higher-order moment on the optimal solution \( w(n) \) of the optimisation problem

\[
\begin{align*}
\min_{w} & \sum_{k=2}^{n} a_k \lambda_k(w) \\
\text{s.t.} & \quad e' \mu = R
\end{align*}
\]

which is the \( n \)-th order approximation to the optimal solution \( w^* \) of

\[
\begin{align*}
\min_{w} & \quad \mathcal{R}_\phi(w'r) \\
\text{s.t.} & \quad e' \mu = R
\end{align*}
\]

where the objective function is the true spectral risk measure. The result in Proposition 4.3 shows that as \( n \) increases, the objective function in (8.3) approximates better \( \mathcal{R}_\phi(w'r) \) for each fixed \( w \) implying that the efficient frontier generated by (8.3) converges to the efficient frontier of (8.4) when \( n \to \infty \). As a consequence, the sequence \( w^{*[2]}, w^{*[3]}, \ldots \) provides an intuition to what degree higher-order moment risk is taken into account.

This analysis is very useful for two reasons. First, it can be used to verify out-of-sample back-tests if taking into account higher-order moment risk results in a significant risk premium. Second, it can provide feedback so that the risk measure can be further customised for particular asset classes or types of investors by additionally tuning the higher-order moment sensitivities.

In order to illustrate the idea, we consider a very simple example of two hedge funds indices — CTA Global and Fixed Income Arbitrage. Considering only two variables allows plotting the objective in (8.3) as a function of \( n \). The same analysis can be repeated on a larger set of asset classes by looking only at how the optimal solution changes with \( n \). The data consists of monthly returns in the period from January 2008 to September 2010. We do not assume any parametric model for
the bivariate return distribution and use the historical data to estimate the L-moments appearing in the expansion.

The distributional characteristics as captured by the first four L-moments are provided in Table 1. The table shows the normalised skewness and kurtosis measures for comparison between the two indices.8 The numbers indicate a significant skewness and kurtosis for FI Arbitrage.

We consider two examples with an exponential and a power spectral risk measure for which the expected return constraint in (8.3) is removed in order to facilitate the example. Figure 7 shows the objective function in (8.3) as a function of \( w_1 \) which denotes the weight of CTA Global for an exponential spectral risk measure with \( \gamma = 8 \). The investment in FI Arbitrage equals \( 1 - w_1 \). The optimal weight changes in the following way: \( w_1^* = 49.5\% \), \( w_1^{*3} = 62.7\% \), \( w_1^{*4} = 62.56\% \), and finally \( w_1^* = 61.06\% \) is the optimal weight for the exponential spectral risk measure. Including the skewness measure in the expansion is already sufficient to approximate the true optimal solution. Naturally, the weight \( w_1^{*3} \) increases since CTA Global has almost zero skewness.

Table 1: The first four L-moments describing the mean, scale, skewness, and the kurtosis of CTA Global and FI Arbitrage hedge fund indices.

<table>
<thead>
<tr>
<th>L-moment</th>
<th>CTA Global</th>
<th>FI Arbitrage</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>0.0054</td>
<td>0.0034</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>0.0127</td>
<td>0.0118</td>
</tr>
<tr>
<td>( \lambda_3 / \lambda_2 )</td>
<td>0.0472</td>
<td>-0.305</td>
</tr>
<tr>
<td>( \lambda_4 / \lambda_2 )</td>
<td>0.0708</td>
<td>0.3389</td>
</tr>
</tbody>
</table>

Figure 7: The corresponding approximations to the exponential spectral risk measure with \( \gamma = 8 \) as a function of the weight of CTA Global.

This result indicates that if the kurtosis as measured by \( \lambda_4 \) is of any concern for the two-fund example, then either a higher value for the risk aversion parameter \( \gamma \) has to be chosen, or a different spectral risk measure has to be chosen. Note that the values of the third-order approximation are not very close to the values of the spectral risk measure for all \( w_1 \). This implies that including more variables may lead to a more significant deviation between \( w_1^{*3} \) and \( w_1^* \).

Figure 7 shows the following information for the power spectral risk measure with \( \alpha = 1.5 \). The optimal weight changes in the following way: \( w_1^{*2} = 49.5\% \), \( w_1^{*3} = 39.6\% \), \( w_1^{*4} = 44.34\% \), and finally \( w_1^* = 44.34\% \). The optimal weight \( w_1^{*2} \) is one and the same irrespective of the choice of the spectral risk measure because the only difference in the objective functions is a positive multiplier of \( \lambda_2 \) which does not change the optimal allocation. In contrast to the exponential spectral risk measure, \( w_1^{*3} \) decreases. The explanation for this phenomenon is available on Figure 3. The positive sensitivity with respect to \( \lambda_3 \) in the expansion leads to this effect. This implies that the choice \( \alpha = 1.5 \) leads to counterintuitive results and needs to be increased if we would like to have intuitive results in the optimal allocation.

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7 - The data is downloaded from the web-site of EDHEC-Risk Institute.
8 - See Hosking (2005) for additional information about the properties of the normalised quantities \( \lambda_3 / \lambda_2 \) and \( \lambda_4 / \lambda_2 \). For the normal distribution, \( \lambda_3 = 0 \) and \( \lambda_4 / \lambda_2 = 3 \tan^{-1} \sqrt{2} - 9 \approx 0.1226 \).
As far as portfolio construction is concerned, there is a trade-off between dimensionality and the value added by the ordinary higher-order moments. The explanation for the trade-off provided by Martellini and Ziemann (2010) is related to the fact that number of parameters increase dramatically leading to a deteriorating impact of the noise in the sample. They demonstrate that improving the estimators can lead to better performance when the universe is not very large.

The same type of trade-off is expected to exist in the L-moment expansion and techniques similar to the ones adopted by Martellini and Ziemann (2010) can be used to improve estimators. The mean-Gini analysis and a corresponding capital asset pricing framework is developed by Shalit and Yitzhaki (1984) as an alternative to the classical mean-variance analysis and linear factor models with the Gini measure are considered by Olkin and Yitzhaki (1992) and Schechtman and Yitzhaki (2007). In contrast to the classical higher-order moments, the L-moments are statistically more robust by definition, therefore the sample noise is expected to be reduced and the value added is expected to be higher.

9. Conclusion
Starting from intuitive geometric characteristics and building up a coherent risk measure is very attractive from both a theoretical and a practical viewpoint. From a risk measurement perspective, such a decomposition provides a link between higher-order moment risk and downside risk in the sense that under some assumptions, they can both be equivalent. It also allows for identifying spectral risk measures with non-intuitive sensitivities to higher-order moments.

From the standpoint of portfolio theory, the counterpart of mean-variance analysis is the mean-Gini analysis. In contrast to mean-variance, however, the L-moment expansion allows for adding measures of skewness and kurtosis to the objective without violating the axioms of coherency. Further on, the expansion can be used to study the incremental value-added in optimal solutions of mean-risk analysis by including incrementally higher-order moments in the objective.

From a practical viewpoint, the construct offers easy interpretations and a constructive method for choosing a risk aversion function depending on the desired higher-order moment sensitivities. Having selected the sensitivities, the calculation of the risk measure boils down to multiplying the sensitivities to the corresponding L-moments and adding them together which can be performed in a spreadsheet. Through the L-moment expansion, the computational complexity of estimating the spectral risk measure is actually transferred to the estimation of the corresponding L-moments which is a more tractable problem.
References


