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The opinions expressed in this study are those of the authors and do not necessarily reflect those of EDHEC Business School. The authors can be contacted at research@edhec-risk.com.
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About the Authors

Noël Amenc is professor of finance and director of development at EDHEC Business School, where he heads the EDHEC-Risk Institute. He has a masters degree in economics and a PhD in finance and has conducted active research in the fields of quantitative equity management, portfolio performance analysis, and active asset allocation, resulting in numerous academic and practitioner articles and books. He is a member of the editorial board of the Journal of Portfolio Management, associate editor of the Journal of Alternative Investments, member of the advisory board of the Journal of Index Investing, and member of the scientific advisory council of the AMF (French financial regulatory authority).

Felix Goltz is head of applied research at EDHEC-Risk Institute and director of research and development at EDHEC-Risk Indices & Benchmarks. He does research in empirical finance and asset allocation, with a focus on alternative investments and indexing strategies. His work has appeared in various international academic and practitioner journals and handbooks. He obtained a PhD in finance from the University of Nice Sophia-Antipolis after studying economics and business administration at the University of Bayreuth and EDHEC Business School.

Stoyan Stoyanov is professor of finance at EDHEC Business School and programme director of the executive MSc in risk and investment management for Asia. He has nearly ten years of experience in the field of risk and investment management. He worked for over six years as head of quantitative research for FinAnalytica. He also worked as a quantitative research engineer at the Bravo Risk Management Group. Stoyan has designed and implemented investment and risk management models for financial institutions, co-developed a patented system for portfolio optimisation in the presence of non-normality, and led a team of engineers designing and planning the implementation of advanced models for major financial institutions. His research focuses on probability theory, extreme risk modelling, and optimal portfolio theory. He has published nearly thirty articles in academic journals, contributed to many professional handbooks, and co-authored two books on financial risk assessment and portfolio optimisation.
Abstract

Since the global financial crisis of 2008, improving risk management practices—management of extreme risks, in particular—has been a hot topic. The postmodern quantitative techniques suggested as extensions of mean-variance analysis, however, exploit diversification as a general method. Although diversification is most effective in extracting risk premia over reasonably long investment horizons and is a key component of sound risk management, it is ill-suited for loss control in severe market downturns. Hedging and insurance are better suited for loss control over short horizons. In particular, dynamic asset allocation techniques deal efficiently with general loss constraints because they preserve access to the upside. Diversification is still very useful in these strategies, as the performance of well-diversified building blocks helps finance the cost of insurance strategies.
Introduction
Introduction

Risk management practices became a central topic after the financial crisis of 2008. Improvements to the methods of risk measurement, many of them made by industry vendors, have drawn on the literature on the modelling of extreme events (Dubikovsky et al. 2010; Zumbach 2007). Although there has been extensive research into extreme risk modelling in academe since the 1950s, it is only after difficult times that the financial industry becomes more open to alternative methods.1

From an academic perspective, however, risk management decision making goes beyond risk measurement and static asset allocation techniques. In fact, it can be argued that the non-classical methods are designed to use two basic techniques in finance—diversification and hedging—in a better way, and with the recent focus on post-modern quantitative techniques the role of diversification as a risk management tool has been over-emphasised. Even though it is a powerful technique, diversification has limitations that must be understood if unrealistic expectations for the real-world performance of risk management are to be avoided.

Although the idea behind it has long existed, a scientifically consistent framework for diversification, modern portfolio theory (MPT), was first posited by Markowitz (1952). Diversification—international diversification, sector and style diversification, and so on—has since become the pillar of many investment philosophies. It has also become a very important risk management technique, so much so that it is often considered, erroneously, synonymous with risk management. In fact, diversification as a general method is related to risk reduction as much as it is to improving performance and, therefore, it is most effective when it is used to extract risk premia. In short, it is only one form of risk management.

The limitations of diversification stem from its relative ineffectiveness in highly correlated environments over relatively shorter horizons. Christoffersen et al. (2010) conclude that the benefits of international diversification across both developed and emerging markets have decreased because of a gradual increase in the average correlation of these markets. Thus, if international markets are well integrated, there is no benefit in diversifying across them.

The variations of correlation are important not only across markets but also over time; in the short run, then, relying on diversification alone can be dangerous. Over longer horizons, Jan and Wu (2008) argue that diversified portfolios on the mean–variance efficient frontier outperform inefficient portfolios, an argument that adds to the debate that time alone may not diversify risks.

The limitations of diversification mean that, in certain market conditions, it can fail dramatically. Using a conditional correlation model, Longin and Solnik (2001) conclude that correlations of international equity markets increase in bear markets. In severe downturns, then, diversification is unreliable. Furthermore, it is generally incapable of dealing with loss control. So enhancing the quantitative techniques behind it by using more sophisticated risk measures and distributional models can lead to more effective diversification but not to

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1 - See, for example, the discussion in Sheikh and Qiao (2009) about a framework for static asset allocation based on non-classical models.
2 - Longin and Solnik (2001) base their model on extreme value theory. There are other studies drawing similar conclusions through models based on other statistical techniques.
Introduction

substantially smaller losses in crashes. Loss control can be implemented in a sound way only by going beyond diversification to hedging and insurance, two other approaches to risk management.

A much more general and consistent framework for risk management is provided by the dynamic portfolio theory posited by Merton (1969, 1971). The theory presents the most natural form of asset management, generalising substantially the static portfolio selection model developed by Markowitz (1952). Merton (1971) demonstrated that in addition to the standard speculative motive, non-myopic long-term investors include intertemporal hedging demands in the presence of a stochastic opportunity set. The model has been extended in several directions: with stochastic interest rates only (Lioui and Poncet 2001; Munk and Sørensen 2004), with a stochastic, mean-reverting equity risk premium and non-stochastic interest rates (Kim and Omberg 1996; Wachter 2002), and with both variables stochastic (Brennan et al. 1997; Munk et al. 2004).

In addition to these developments, recognising that long-term investors usually have such short-term constraints as maximum-drawdown limits, or a particular wealth requirement, leads to further extensions of the model. Minimum performance constraints were first introduced in the context of constant proportion portfolio insurance (CPPI) (Black and Jones 1987; Black and Perold 1992), and in the context of option-based portfolio insurance (OBPI) (Leland 1980). More recent papers (Grossman and Zhou 1996) demonstrate that both of these strategies can be optimal for some investors and subsequent papers generalise the model by imposing minimum performance constraints relative to a stochastic, as opposed to a deterministic, benchmark. Teplá (2001), for example, demonstrates that the optimal strategy in the presence of such constraints involves a long position in an exchange option.4

The much more general and flexible dynamic portfolio theory leads to new insight into risk management in general and the role of diversification. In this framework, diversification provides access to performance through a building block known as a performance-seeking portfolio (PSP). Downside risk control is achieved by assigning state-dependent—and possibly dynamic—weights to the PSP and to a portfolio of safe, or risk-free, assets.

In fact, since the latest financial crisis, there has been confusion among market participants not only about the benefits and limitations of diversification as a method for risk management but also about how the methods of hedging and insurance are related to diversification. In this paper, our goal is to review diversification and clarify its purpose. Going back to the conceptual underpinnings of several risk management strategies, we see that, in a dynamic asset management framework, diversification, hedging, and insurance are complementary rather than competing techniques for sound risk management. The paper is organised in two parts. The first discusses the benefits and limits of diversification. The second moves on to hedging and insurance and discusses diversification as a method of reducing the cost of insurance.
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1. Advantages and Disadvantages of Diversification
1. Advantages and Disadvantages of Diversification

Diversification and mean-variance analysis

Diversification is one of the most widely used general concepts in modern finance. The principle can be traced back to ancient times, but as far as portfolio construction is concerned, the old saw about not putting all your eggs in one basket captures the essence of the approach on a more abstract level—reduce portfolio concentration to improve its risk/return profile.

Portfolio concentration can be reduced in a number of different ways, from ad hoc methods such as applying equal weights to methods based on solid scientific arguments. A landmark publication by Markowitz (1952) laid the foundations for a scientific approach to optimal distribution of capital in a set of risky assets. The paper introduced mean-variance analysis and demonstrated that diversification can be achieved through a portfolio construction technique that can be described in two alternative ways: (i) maximise portfolio expected return for a given target for variance or (ii) minimise variance for a given target for expected return. The portfolios obtained in this fashion are called efficient and the collection of those portfolios in the mean-variance space is called the efficient frontier. Therefore, conceptually, the mean-variance analysis links diversification with the notion of efficiency—optimal diversification is achieved along the efficient frontier.

The principles behind the Markowitz model can be formalised in the following optimisation problem

\[
\begin{align*}
\min_w&w' \Sigma w \\
\text{s.t. } &w' e = 1 \\
&w' \mu = m 
\end{align*}
\]

where \( \Sigma = \{ \text{cov}(R_i, R_j) \}_{i,j} \) is the covariance matrix of stock returns, \( w = (w_1, \ldots, w_n) \) is the vector of portfolio weights, \( \mu \) is a vector of expected returns, \( m \) is the target portfolio return, and \( e = (1, \ldots, 1) \). The objective function is in fact portfolio variance, the first constraint states that portfolio weights should add up to one and the second constraint sets the portfolio return target.

The optimisation problem in Eq. 1 implies that there are three important inputs—the standalone characteristics represented by the vector of expected returns and the variance of stock returns positioned on the main diagonal of the covariance matrix, as well as the joint behaviour of stock returns represented by the covariance collected in the off-diagonal elements of \( \Sigma \). The last input leads to a very important insight indicating that joint behaviour is crucial to the notion of efficient portfolios; it explains why diversification works.5

In fact, one limitation of the method can be identified by recognising that diversification is less effective when asset returns are more highly correlated. This conclusion follows from the decomposition of portfolio variance into two terms

\[
\sigma_p^2 = w' \Sigma w = \sum_i w_i^2 \sigma_i^2 + \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \rho_{ij}
\]

where \( \rho_{ij} = \text{corr}(R_i, R_j) \) is the corresponding correlation coefficient. The second term is the contribution of correlation to total portfolio variance. If \( \rho_{ij} \) is close to 1 for all assets, then there is a single factor driving the returns of all assets. Therefore, distributing capital among many assets is just as effective as investing in one asset.

---

5 - If joint behaviour were unimportant, investing 100% of the capital in the least risky stock would always represent the least risky portfolio.
1. Advantages and Disadvantages of Diversification

only. More formally, if all correlations are exactly equal to 1, total portfolio variance can be represented as

\[ \sigma^2_p = (w_1 \sigma_1 + \ldots + w_n \sigma_n)^2 \]

meaning that without a return target the optimal solution to Eq. 1 is a 100% allocation to the least risky asset. In this situation, diversification is ineffective since the optimal solution is a totally concentrated portfolio.\(^6\)

From an investor perspective, solving the problem in Eq. 1 means optimising the risk/return tradeoff because risk is minimised conditional on a return target. As a result, diversification as a general method is not only about risk reduction. In fact, assuming the opposite would imply that the most diversified portfolio is the global minimum variance (GMV) portfolio, which is obtained by dropping the second constraint in Eq. 1. This statement is arguable, however, as GMV portfolios can be concentrated on the relatively lower-volatility stocks, which also implies concentration in such sectors as utilities.\(^7\) In fact, building well-diversified portfolios is more about efficient extraction of risk premia than about mere risk minimisation. This conclusion, however, assumes that diversification is designed to work over the long run across different market conditions. Along with the influence of correlation on diversification opportunities, this assumption is another drawback of the approach.

In a market crash, for example, asset returns become highly correlated and the shortcomings of diversification are highlighted. This empirical result is illustrated in figure 1, in which we show the average correlation of the sector indices of the S&P 500 from the beginning of 2000 to 2010. The average correlation increased around the dot-com bubble and the 9/11 attacks and in the financial meltdown of 2008.

\[ \text{Figure 1: The average correlation of the sectors in the S&P 500 calculated over a two-year rolling window} \]

In these conditions, as illustrated in figure 2, in which we compare the in-sample performance of two optimised strategies—the maximum Sharpe ratio (MSR) and the GMV portfolios—to that of the equally weighted (EW) portfolio and the cap-weighted S&P 500, diversification is unhelpful. In all cases, the universe consists of the sector indices of the S&P 500. The plot shows that all strategies, even the optimised ones, post large losses during the crash of 2008. These losses are reflected in table 1, which shows the maximum-drawdown statistics for the strategies in the period between January 2007 and September 2010.

\[ \text{Table 1: The maximum drawdown experienced by the strategies in figure 2 between January 2007 and September 2010} \]

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Max drawdown</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSR</td>
<td>24.33%</td>
</tr>
<tr>
<td>GMV</td>
<td>24.45%</td>
</tr>
<tr>
<td>EW</td>
<td>49.43%</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>52.56%</td>
</tr>
</tbody>
</table>

\(^6\) We assume that the portfolio is long-only. If unconstrained shorting is allowed, then it is possible to construct a zero-volatility portfolio from any pair of perfectly positively correlated assets having different volatilities. Since risk can be hedged completely using only two assets, it follows that there is no point in building a diversified portfolio under these assumptions as well.\(^7\) See appendix 1 for a theoretical remark on the structure of GMV portfolios.
1. Advantages and Disadvantages of Diversification

Figure 2: Even though optimised portfolios such as MSR and GMV are well diversified, they suffered large losses during the 2008 crisis. For comparison, the EW portfolio and the cap-weighted S&P 500 are also shown.

There are, however, good reasons for the failure of diversification to reduce losses in sharp market downturns. Increased correlation, common in downturns, limits diversification opportunities. Perhaps more importantly, diversification is designed to extract risk premia in an efficient way over long horizons, not to control losses over short horizons. Misunderstanding the limitations of the approach can mislead investors into concluding that, since diversification did not protect them from big losses in 2008, it is a useless concept.

Diversification and general alternative risk models

Even though diversification is a generic concept, we use mean-variance analysis to exemplify its advantages and disadvantages. Mean-variance analysis is based on the assumption that risk-averse investors maximise their expected utility at the investment horizon and take into account only two distributional characteristics—mean and variance. This assumption is realistic either if asset returns are normally distributed or if investors have quadratic utility functions; both of which assumptions are overly simplistic. Empirical research has firmly established that—especially at high frequencies—asset returns can be skewed, leptokurtic, and fat-tailed and quadratic utility functions arise in the model as a second-order Taylor series approximation of a general utility function.

Using variance as a proxy for risk is also controversial. A disadvantage often pointed out is that it penalises losses and profits symmetrically while risk is an asymmetric phenomenon associated more with the left tail of the return distribution. Therefore, a realistic risk measure would be more sensitive to the downside than to the upside of the return distribution. At a given confidence level $\alpha$, Value-at-Risk (VaR), a downside risk measure widely used in the industry, is implicitly defined as a threshold loss such that the portfolio loses more than VaR with a probability equal to $1 - \alpha$

$$P(X \leq VaR_\alpha) = 1 - \alpha$$

where $X$ is a random variable describing the portfolio return distribution.

Since diversification as a concept goes beyond mean-variance analysis, it has been argued that failure in market crashes is caused mainly by the inappropriate assumptions made by the Markowitz model. If a downside risk measure is used instead of variance, the portfolio may perform better during severe crashes. Which downside risk measure is appropriate, however, is not clear and VaR is hardly the only alternative.

Although different ways of measuring risk have been discussed since the 1960s,
1. Advantages and Disadvantages of Diversification

An axiomatic approach was taken in the 1990s\textsuperscript{10} with the development of firm-wide risk measurement systems. The first axiomatic construction was that of coherent risk measures by Artzner \textit{et al.} (1998). The axiom that guarantees that diversification opportunities would be recognised by any coherent risk measure is that of sub-additivity,

\textbf{Eq. 3} \quad \rho (X + Y) \leq \rho (X) + \rho (Y)

where \(\rho\) denotes the measure of risk and \(X\) and \(Y\) are random variables describing the returns of two assets, \textit{i.e.}, the risk of a portfolio of assets is less than or equal to the sum of the risks of the assets. It is possible to reformulate the portfolio selection problem in Eq. 1 with any risk measure satisfying Eq. 3 in the objective function; that is, instead of minimising variance, we can minimise a sub-additive risk measure subject to the same constraints.

An axiomatic approach, however, implies that there could be many risk measures satisfying the axioms, and sub-additivity axiom in particular. As a consequence, the choice of a particular risk measure for the portfolio construction problem becomes difficult and must be made on the basis of additional arguments. Standard deviation, for example, satisfies the sub-additivity axiom. This conclusion is apparent from equation Eq. 2—the second term, which involves the correlations, is the reason sub-additivity holds. VaR is generally not sub-additive, but it is robust, easy to interpret, and required by legislation and, as a consequence, it is widely used. Furthermore, recent research\textsuperscript{11} indicates that sub-additivity holds when the confidence level is high enough and the returns are fat-tailed. A risk measure suggested as a more informative, coherent (and therefore sub-additive) alternative to VaR is Conditional Value-at-Risk (CVaR). It measures the average loss as long as the loss is larger than the corresponding VaR.

We are interested in whether or not adopting a downside risk measure results in dramatically different performance in market crashes.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The in-sample performance of GM CVaR and GM VaR portfolios, both risk measures at the 95\% confidence level, during the crash of 2008, together—for comparison—with the cap-weighted S&P 500.}
\end{figure}

\begin{table}
\centering
\begin{tabular}{|l|c|}
\hline
\textbf{Strategy} & \textbf{Max drawdown} \\
\hline
GM CVaR & 22.92\% \\
GM VaR & 29.15\% \\
\hline
\end{tabular}
\caption{The maximum drawdown experienced by the strategies in figure 3 between January 2007 and September 2010.}
\end{table}

Although using a downside risk measure may help fine-tune the benefits of diversification, it clearly does not help much in severe market downturns. Figure 3 and table 2 provide an illustration for the period from January 2007 to September 2010, the same period as that in figure 2. Since the point of this illustration is to compare results in times of large market downturns, we limit the comparison to this time period only.

\textsuperscript{10} Markowitz (1959) suggested semi-variance as a better alternative to variance as a proxy for risk, as it concerns only adverse deviations from the mean.

\textsuperscript{11} See Danielsson \textit{et al.} (2010).
1. Advantages and Disadvantages of Diversification

Holding everything else equal, we consider CVaR and VaR alternative risk measures at a standard confidence level of 95% for both. Figure 3 shows the values of the global minimum CVaR (GM CVaR) and the global minimum VaR (GM VaR) portfolios through time and table 2 shows the corresponding maximum-drawdown statistics. The losses in table 2 are significant, though the GM CVaR portfolio leads to drawdown marginally lower that that of the GMV portfolio (see table 1).

That table 2 shows no significant reduction in drawdown is unsurprising. By building the GM VaR portfolio, we are actually minimising the loss occurring with a given probability (5% in the example in figure 3). There is no guarantee that large losses will not be observed. Likewise, by building the GM CVaR portfolio, we are minimising an average of the extreme losses. Again, having a small average extreme loss does not necessarily imply an absence of large losses in market crashes.

In fact, it is possible to make a more general statement that is independent of the choice of risk measure. In the previous section, we argue that diversification opportunities disappear when the correlation of asset returns is close to 1. Leaving the multivariate normal world complicates the analysis, but it is possible to demonstrate that diversification opportunities disappear if asset returns become comonotonic (increasing functions of each other), which corresponds to perfect linear dependence in the Markowitz framework.

In Eq. 3 the joint distribution of X and Y can be any; the property is assumed to hold for all possible multivariate distributions and, by design, for all coherent risk measures. As a result, the dependence structure of the asset returns determines the presence of diversification opportunities, whereas the function of the risk measure is to identify them and transform them into actual allocations. For the worst possible dependence structure, which is that of functional dependence, the inequality in Eq. 3 turns into an equality, which means that it is not possible to find a portfolio whose risk is smaller than the weighted average of the standalone risks. Intuitively, under these circumstances, a 10% drop in one of the assets determines exactly the changes in the other assets, since they are increasing functions of each other. In a situation such as this one, holding a broadly diversified portfolio is just as good as holding only a few assets.

As a consequence, we can argue that generalising the mean-variance framework leads to the conclusion that, if securities are nearly functionally dependent in market crashes, then there are no diversification opportunities. Under these conditions, choosing a risk measure is redundant because the argument is generic (see appendix 2 for additional details).

Statistical arguments provide evidence for this conclusion as well. Figures 2 and 3 show the in-sample performance of the optimised strategies. In this calculation, we assume perfect knowledge of the mean and variance in the Markowitz analysis and perfect knowledge of the multivariate distribution for the GM CVaR and GM VaR examples. Yet in these perfect conditions, none of the optimised strategies is able to provide reasonable loss protection in 2008. In reality, the optimal solutions would be

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12 - See, for example, Ekeland et al. (2009) and Rüschendorf (2010).
13 - Comonotonicity is in fact a characteristic of the upper Frechet-Hoeffding bound of any multivariate distribution. Since in this analysis we hold the marginal distributions fixed, it follows that the comonotonic behaviour is a property of the dependence structure of the random vector, or the so-called copula function. As a consequence, the presence of diversification opportunities is a copula property. This statement is in line with the conclusion that diversification opportunities are a function of correlations in the Markowitz framework since the copula function in the multivariate Gaussian world is uniquely determined by the correlation matrix.
14 - We need the technical condition \(\sup_{\rho (X,Y)} \rho(X+Y) = \rho(X) + \rho(Y)\) where the supremum is calculated over all bivariate distributions \((X,Y)\) with fixed marginals. This condition is introduced as a separate axiom in Ekeland et al. (2009). See appendix 2 for additional details.
1. Advantages and Disadvantages of Diversification

influenced by the noise coming from our imperfect knowledge of these parameters, suggesting that the results may be even worse. However, our results with perfect parameter knowledge show that attempts to improve the parameter estimators, or the model for the multivariate distribution, will be of little help in reducing the drawdown of optimally diversified portfolios in severe market crashes.

**Diversification and higher-order comoments**

Another way to extend the framework beyond the mean-variance analysis is to consider higher-order Taylor series approximations of investor’s utility function. The higher-order approximation results in higher-order moments in the objective function of the portfolio optimisation problem given in Eq. 1 (Martellini and Ziemann 2010). Using the fourth-order approximation, for example, means incorporating portfolio skewness and kurtosis in addition to portfolio variance. In this way, the objective function becomes more realistic in the sense that it takes into account the empirical facts that asset returns are asymmetric and exhibit excess kurtosis.

This problem setup makes it possible to identify diversification opportunities other than those available in the correlation matrix because portfolio skewness and kurtosis depend on the coskewness and cokurtosis of asset returns that represent statistical measures of dependence of the asymmetries and the peakedness of the stock return distributions. The coskewness and cokurtosis appear in addition to covariance and describe other aspects of the joint behaviour.

The additional information, however, comes at a cost. The coskewness and cokurtosis parameters increase significantly the total number of parameters that need to be estimated from historical data. Thus, a portfolio of 100 assets would require estimation of more than 4.5 million parameters. Compared to accounting for higher-order moments when coskewness and cokurtosis parameters are estimated without properly handling estimation risk, a simple mean-variance approach thus tends to lead to better out-of-sample results since it avoids the error-prone estimation of higher-order dependencies. Nevertheless, Martellini and Ziemann (2010) demonstrate that, for lower-dimensional problems, if the parameter estimation problem is properly handled, including higher-order comoments adds value to the portfolio selection problem and can lead to higher risk-adjusted returns, indicating that it provides access to additional diversification opportunities. As for protection from losses in extreme market conditions, however, this approach is no more helpful than any of the others discussed in the previous sections.
1. Advantages and Disadvantages of Diversification
2. Beyond Diversification: Hedging and Insurance
2. Beyond Diversification: Hedging and Insurance

The discussion in the previous section illustrates the benefit of diversification, which is to extract risk premia, and two key shortcomings: (i) it is unreliable in highly correlated markets and (ii) it is not an efficient technique of loss control in the short term. Complaints that diversification has failed are somewhat misleading, as it was never meant to provide downside protection in market crashes. From a practical viewpoint, it is important to transcend diversification and to identify techniques that can complement it and offset its shortcomings.

One potential technique is hedging, generally used to offset partially or completely a specific risk. Hedging can be done in a variety of ways; the best example, perhaps, is through a position in futures. Suppose that a given portfolio has a long exposure to the price of oil, a risk the portfolio manager is unwilling to take over a given horizon. One possibility is to enter into a short position in an oil futures contract. If the portfolio has an undesirable long exposure to a given sector (financials, say), another hedging strategy is to short sell the corresponding sector index. Depending on the circumstances, the hedge can be perfect, if the corresponding risk is completely removed, or imperfect (partial), leading to some residual exposure.

In the following section, we discuss the advantages and disadvantages of combining hedging and diversification. The limitations of this combination stem largely from the static nature of hedging. Insurance, which is dynamic in nature—and the second topic of this section—can be used to overcome these limitations.

Hedging: fund separation and risk reduction

The mean-variance framework introduced by Markowitz (1952) does not consider a risk-free asset; the investable universe consists of risky assets only. Tobin (1958), however, argued that, in the presence of a risk-free asset, investors should hold portfolios of only two funds—the risk-free asset and a fund of risky assets. The fund of risky assets is the maximum Sharpe ratio (MSR) portfolio constructed from the risky assets. Furthermore, the risk aversion of investors does not change the structure of the efficient MSR fund; it affects only the relative weights of the two funds in the portfolio. This arrangement is the result of a so-called two-fund separation theorem, which posits that any risk-averse investor can construct portfolios in two steps: (i) build the MSR portfolio from the risky assets and (ii) depending on the degree of risk-aversion, hedge partially the risk present in the MSR portfolio by allocating a fraction of the capital to the risk-free asset.

Figure 4: The in-sample efficient frontier of the risky assets (in blue) and the CML (capital market line) together with the tangency portfolio, the GMV portfolio, and the portfolio with the same risk as the GMV on the CML. The annualised risk-free rate is set to 2%.15

From a geometric perspective, adding a risk-free asset to the investable universe results in a linear efficient frontier called the
2. Beyond Diversification: Hedging and Insurance

capital market line (CML), a line tangential to the efficient frontier generated by the risky assets. Since the point of tangency is the MSR portfolio, it is also known as the tangency portfolio. Figure 4 illustrates the geometric property.\footnote{16 - The risky assets generating the efficient frontier on the plot are the sector indices of the S&P 500. We consider the ten-year period from 2000 to 2010. The weights in the optimisation problem are between -40% and 40%. The risk-free asset is assumed to yield an annual return of 2%, a return representative of the average three-month Treasury bill rate from 2000 to 2010.}

Introducing a risk-free asset and partial hedging as a technique for risk reduction raises the following question. For a given risk constraint, which portfolio construction technique is better? Taking advantage of diversification, maximising expected return subject to the risk constraint and choosing the portfolio on the efficient frontier of the risky assets, or taking advantage of the fund-separation theorem and, instead of building a customised portfolio of risky assets, partially hedging the risk of the MSR portfolio with the risk-free asset to meet the risk constraint? From a theoretical perspective, the second approach is superior because the risk-adjusted return of all portfolios on the CML is not smaller than those on the efficient frontier of the risky assets.

The components of the portfolio account for its better risk/return tradeoff. The efficient MSR portfolio is constructed to provide the highest possible risk-adjusted return. Therefore, it is in the construction of this portfolio that we take advantage of diversification to extract premia from the risky assets. The MSR portfolio is in fact responsible for the performance of the overall strategy. The risk-free asset, by contrast, is there to hedge risk. In fact, the fund-separation theorem implies that there is also a functional separation—the two funds in the portfolio are responsible for different functions.

Although volatility is kept under control, both the GMV portfolio and the GMV match on the CML (see figure 4) post heavy losses in the crash of 2008. Unlike diversification, however, hedging can be used to control extreme losses. In theory, the risk-free asset has universal hedging properties. If the portfolio is allocated entirely to the risk-free asset, then, in theory, it grows at the risk-free rate. Appropriate allocation to the risk-free asset can thus hedge partially all aspects of risk arising from the uncertainty in the risky assets. We can easily, for example, construct a portfolio on the CML with an in-sample maximum drawdown of no more than 10%. For our dataset, it turns out that a portfolio with this property is obtained with a 40% allocation to the MSR portfolio. Explicit loss control of this type is not possible if the investor relies only on diversification.

Figure 5: The performance and the dynamics of the maximum drawdown of the GMV portfolio and the GMV match on the capital market line

To check this conclusion in practice, we choose the GMV portfolio on the efficient frontier of the risky assets and the portfolio with the same risk on the CML. The in-sample performance of the two portfolios is shown in figure 5. Both portfolios are equally risky in terms of volatility but the one on the CML performs better.
## 2. Beyond Diversification: Hedging and Insurance

A comparison of the performance of three portfolios—the GMV portfolio, the GMV match on the CML, and a portfolio on the CML constructed such that it has a maximum drawdown of 10%—is shown in figure 6. Hedging makes it possible to match in-sample any maximum drawdown, regardless of its size. Since the portfolio return distribution is a weighted average of the return distribution of the MSR portfolio and a constant,

$$ r_p = v r_{MSR} + (1 - v) r_f $$

where $0 \leq v \leq 1$ is the weight of the MSR portfolio and $r_f$ the risk-free rate, it follows that by changing $v$ the portfolio return distribution is scaled up or down. Using Chebychev’s inequality, it is possible to demonstrate that the probability of large losses can be made infinitely small by reducing $v$,

$$ P(|r_p - E(r_p)| > e) \leq \frac{\sigma_{MSR}^2}{e^2} $$

in which $\sigma_{MSR}^2$ is the variance of the MSR portfolio. Even though this approach is capable of controlling the downside of the return distribution, there is a caveat. Along with the left tail, scaling influences symmetrically the right tail of the return distribution. As a consequence, this approach can lead to limited drawdown but at the cost of lower upside potential.

### Insurance: dynamic risk management

In the previous example, the reason for the lower upside potential is the fact that hedging is a static technique. The entire analysis takes place in a single instance and the optimal portfolio is, essentially, a buy-and-hold strategy. As a consequence, the weight of the MSR does not depend on time or on the state of the market. Ideally, investors would demand an improved downside and an improved upside at the same time. This, however, is not feasible with a static technique.

Simple forms of dynamic risk management, also called portfolio insurance, were suggested in the late 1980s. Black and Jones (1987) and Black and Perold (1992) were the first to suggest constant proportion portfolio insurance (CPPI). This strategy is a dynamic trading rule that allocates capital to a risky asset and cash in proportion to a cushion that is the difference between the current portfolio value and a selected protective floor. The resulting payoff at the horizon is option-like because the exposure to the risky asset approaches zero if the value of the portfolio approaches the floor. The overall effect is similar to that of owning a put option—CPPI guarantees that the floor will not be breached.

Another popular insurance strategy is option-based portfolio insurance (OBPI) (Grossman and Vila 1989). This strategy basically consists of buying a derivative instrument so that the left tail of the payoff distribution at the horizon is truncated at a desired threshold.
2. Beyond Diversification: Hedging and Insurance

The derivative instrument can be a simple European call option or an exotic product depending on additional path-wise features we would like to engineer.

Even though CPPI and OBPI are conceptually simple, they seem to be based on separate techniques rather than on a more basic framework. Nevertheless, since the option can, in theory, be replicated dynamically, both CPPI and OBPI can be viewed as members of a single family of models. In fact, a much more general extension is valid. The dynamic portfolio theory developed by Merton (1969, 1971) can be extended with absolute or relative constraints on asset value and it is possible to show that both CPPI and OBPI arise as optimal strategies for investors subject to particular constraints (Basak 1995, 2002).

The treatment of the constraints in continuous-time dynamic portfolio theory is generic; they are introduced in terms of a general floor. The floors can be absolute or relative to a benchmark portfolio. An absolute floor, for instance, can be any of the following:

- **A capital-guarantee floor.** The floor is calculated by the formula
  
  \[ F_t = k e^{-\gamma(T-t)} A_0 \]

  where \( r_f \) is the risk-free rate, \( T-t \) calculates the time to horizon, \( A_0 \) is the initial portfolio wealth, and \( k < 1 \) is a positive multiplier. This floor is usually used in CPPI and non-violation of this floor guarantees that the strategy will provide the initial capital at the horizon.

- **A rolling-performance floor.** This floor is defined by
  
  \[ F_t = A_t - t^* \]

  where \( t^* \) is a predefined period of time, twelve months, for example. The rolling-performance floor guarantees that the performance will stay positive over period \( t^* \).

- **A maximum-drawdown floor.** A drawdown constraint is implemented by
  
  \[ F_t = \alpha \max_{\text{ret}} A_t \]

  where \( \alpha \) is a positive parameter less than 1 and \( A_t \) is portfolio wealth at time \( t \). A maximum-drawdown floor implies that the value of the portfolio never falls below a certain percentage, \( 100(1 - \alpha)\% \), of the maximum value attained in the past. This constraint was initially suggested as an absolute constraint but can be reformulated as a relative one.

- **A relative-benchmark floor.** This relative floor is defined by
  
  \[ F_t = kB_t \]

  where \( k < 1 \) is a positive multiplier and \( B_t \) is the value of a benchmark at time \( t \). This floor guarantees that the value of the portfolio will stay above \( 100k\% \) of the value of the benchmark.

Several floors can be combined together in a single floor by calculating their maximum,

\[ F_{\text{max}} = \max\{F_0, ..., F_t\} \]

The new floor \( F_{\text{max}} \) can then be adopted as a single floor in the dynamic portfolio optimisation problem. It follows from the definition that if \( F_{\text{max}} \) is not violated, then none of the other floors will be, either.
2. Beyond Diversification: Hedging and Insurance

Solving a dynamic asset allocation problem with an implicit floor constraint results in an optimal allocation of the following form,

\[
    w^* = \frac{1}{\gamma} \left( 1 - \frac{F_t}{A^*_t} \right) P_{SP} + \left( 1 - \frac{1}{\gamma} \left( 1 - \frac{F_t}{A^*_t} \right) \right) S_{AFE}
\]

where \( P_{SP} \) is the generic notation for the weights of a performance-seeking portfolio, \( S_{AFE} \) the weights in the safe assets, \( \gamma \) the degree of risk aversion, \( F_t \) the value of the selected floor at time \( t \), and \( A^*_t \) the value of the optimal constrained portfolio (see appendix 3 for additional details).

The solution in Eq. 5 is a fund-separation theorem in a dynamic asset allocation setting. The optimal weight equals a weighted average of two building blocks constructed for different purposes. The PSP is constructed for access to performance through efficient extraction of risk premia; in fact, under fairly general assumptions it is the MSR portfolio.

The general goal of the SAFE building block is to hedge liabilities. In the very simple example of the previous section, SAFE consists of a government bond maturing at the investment horizon. In a dynamic setting, depending on the institution constructing the strategy, SAFE has a different structure. For example, critical factors for pension funds are interest rates and inflation. As a result, the SAFE portfolio for a pension fund would contain assets hedging interest rate risk and inflation risk (see appendix 3 for additional details).

Even though Eq. 5 is much more general than Eq. 4, considering only the building blocks, the greatest difference is in the weights of the building blocks. In Eq. 4, the weights are static, whereas in Eq. 5 they are state- and, potentially, time-dependent. This is the improvement that makes insurance an adequate general approach to downside risk management.

An illustration of the improvement of insurance strategies on hedging is provided in figure 7. In the upper part of the figure, we compare the in-sample performance of the 10% maximum-drawdown strategy obtained through the static methods of hedging and a dynamic strategy with a maximum-drawdown floor of 10%. The lower part of the figure shows a plot of the dynamics of the allocation to the MSR portfolio and illustrates how insurance strategies control downside losses. When there is a market downturn and the value of the portfolio approaches the floor, the allocation to the PSP building block, or the MSR portfolio in this case, decreases. When the value of the portfolio hits the floor, as it nearly does in the crash of 2008 (see figure 7), allocation to the risky MSR portfolio stops altogether and the portfolio is totally invested in the SAFE building block. Since the SAFE asset is supposed to carry no risk, it is not possible, in theory, to breach
2. Beyond Diversification: Hedging and Insurance

The drawdown characteristics of the two strategies are shown in Table 3. To all appearances, they both exhibit similar in-sample average and maximum drawdown. The dynamic strategy, however, has greater upside potential, a result of the design of the MSR portfolio.

The difference in the properties of the static and the dynamic approaches are best illustrated in a Monte-Carlo study. Figure 7 compares the performance of only two paths, but in practice we need more than two to gain insight into the difference in the extreme risk exposure of the two strategies. We fitted a geometric Brownian motion (GBM) to the MSR sample path and generated 5,000 paths with a ten-year horizon. For each path, which represents one state of the world in this setting, we calculated the dynamic strategy with a 10% maximum drawdown. The static strategy is a fixed-mix portfolio with a 40% allocation to the MSR portfolio. Then, in each state of the world, we calculated the annualised returns and the maximum drawdown of the two strategies.

The results are summarised in Figure 8. The plot on the left shows the histograms of the annualised return distribution for the two strategies superimposed. The blue histogram indicates better access to the upside performance of the dynamic strategy. The plot on the right shows the corresponding histograms for the maximum-drawdown distribution. The great difference stems from the inability of the static approach to keep losses under control. In some states of the world, the maximum drawdown reaches more than 20%, even though the same static strategy was designed to have a 10% in-sample maximum drawdown. In contrast, there is no single state of the world in which the dynamic strategy has a maximum drawdown greater than 10%.

---

21 In a practical implementation, a breach of the floor may occur because, as a result of turnover constraints, trading may need to be less frequent, which can result in a breach occurring between rebalancing dates, or because a perfect hedge with the SAFE portfolio may not be possible as a result of market incompleteness, which implies that there may be residual risks in the portfolio. Nevertheless, dynamic asset allocation is the right general approach to controlling downside risks.

---

Table 3: The maximum and average drawdown of the two strategies in figure 7 between January 2007 and September 2010

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Average drawdown</th>
<th>Max drawdown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic strategy</td>
<td>2.9%</td>
<td>9.2%</td>
</tr>
<tr>
<td>Static strategy</td>
<td>2%</td>
<td>10%</td>
</tr>
</tbody>
</table>
2. Beyond Diversification: Hedging and Insurance

The risk-return characteristics calculated from the distributions shown in figure 8 are shown in table 4. The annualised average return of the dynamic strategy is higher than that of the static strategy, as expected, and the big difference in the maximum-drawdown distributions is apparent. The average maximum drawdown of the static strategy is near the in-sample figure of 10%.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Annualised average return</th>
<th>Average max drawdown</th>
<th>Largest max drawdown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic strategy</td>
<td>9.56%</td>
<td>8%</td>
<td>9.64%</td>
</tr>
<tr>
<td>Static strategy</td>
<td>8.26%</td>
<td>10.32%</td>
<td>28.3%</td>
</tr>
</tbody>
</table>

The risk-return characteristics calculated from the distributions shown in figure 8 are shown in table 4. The annualised average return of the dynamic strategy is higher than that of the static strategy, as expected, and the big difference in the maximum-drawdown distributions is apparent. The average maximum drawdown of the static strategy is near the in-sample figure of 10%.

Diversification and the cost of insurance

Diversification can be implemented, at least in theory, at no cost, but insurance always has a cost. The cost of insurance is easiest to spot in the OBPI strategies in which a certain amount of capital is invested in a derivative instrument. In this case, the cost is the price of the derivative. Since the derivative can usually be replicated by a dynamic portfolio, it is clear that such costs can be present in other types of dynamic insurance strategies. In such cases, however, they materialise as implicit opportunity costs.

One way to illustrate the cost of insurance is to look at the return distribution of the dynamic strategy at the investment horizon and the corresponding histogram of the PSP building block. The opportunity cost of insurance appears as a lower expected return for the dynamic strategy.

We did a Monte-Carlo study to illustrate this effect on an insurance strategy with a maximum-drawdown constraint. The PSP building block is modelled as a GBM,

\[
\text{Eq. 6 } dS_t = S_t(r_f + \sigma \lambda)dt + S_t \sigma dB_t
\]

where \( \lambda \) is the Sharpe ratio of the strategy. We adopt the parameter values calibrated in Munk et al. (2004) and have the Sharpe ratio be \( \lambda = 0.24 \), which corresponds to the long-term ratio for the S&P 500. We generated 5,000 sample paths from the model in Eq. 6 with an investment horizon of ten years. For each sample path, we calculated the dynamic insurance strategy and computed its average annual return, as well as the average annual return of the PSP component.

Figure 9: The return distribution of a dynamic strategy compared to that of the PSP component. The top pair of plots is produced with the default value of \( \lambda = 0.24 \) and the bottom pair of plots is produced with \( \lambda = 0.36 \), which is a 50% improvement on the default value. SP is shortfall probability—the probability that the annualised average return will be negative.
The top pair of plots in figure 9 compares the two distributions. The annualised expected return of the dynamic strategy is 6.72%, whereas that of the PSP building block is 8.46%. Although the difference in the annualised return distribution seems large on the plot, it must be kept in mind that drawdown protection results in good path-wise properties that are hard to spot in the histogram of the dynamic strategy in figure 9. The good path-wise properties materialise as a significantly smaller shortfall probability (SP).

One way to offset the cost of insurance is to improve the building blocks of the dynamic strategy. Since the PSP is devoted to performance, it must be constructed as a well-diversified portfolio. In practice, the common approach is to adopt a standard stock market index, a cap-weighted portfolio.

Although cap-weighted indices are popular in the industry, there is ample empirical evidence that they are poorly diversified and highly inefficient (Haugen and Baker 1991; Grinold 1992; Amenc et al. 2006). The reason is that capitalisation weighting leads to high concentration in a handful of stocks. In fact, equally weighted portfolios, although naïvely diversified, have been found to provide higher risk-adjusted returns.

Although it has been shown that even naïvely diversified portfolios dominate the corresponding cap-weighted portfolios, equal weighting provides optimal diversification from the standpoint of mean-variance analysis if and only if all securities have identical expected returns, volatility, and if all pairs of correlation are the same. Since this hypothesis is highly unrealistic, there is a clear indication that, by carefully estimating the risk and return parameters, it would be possible to construct risk-efficient MSR portfolios providing superior risk-adjusted returns.

Successful implementation of an MSR portfolio is critically dependent on the quality of the parameter estimators. Amenc et al. (2010a) do an empirical study for the S&P 500 universe from January 1959 to December 2008. They show that using parameter estimation techniques resulting in robust estimates of the risk and the return parameters leads to an optimised strategy with a Sharpe ratio more than 50% higher than the Sharpe ratio of the S&P 500 index.

If improving diversification makes possible a 50% improvement in the Sharpe ratio of the PSP, it is interesting to see to what extent...
2. Beyond Diversification: Hedging and Insurance

degree it can offset the implicit cost of insurance. So we regenerated the scenarios from the model in Eq. 6, keeping the same parameter values and increasing the Sharpe ratio to 0.36. The histograms of the return distributions of the dynamic strategy before and after the Sharpe ratio improvement are compared in the bottom pair of plots in figure 9. The annualised expected return of the dynamic strategy improves from 6.72% to 8.19%, a jump that, in this context, implies that improving the Sharpe ratio of the PSP by 50% very nearly compensates for the cost of insurance.
Conclusion
The global financial crisis of 2008 has shifted the attention of all investors to risk management. In a broad context, risk management is about maximising the probability of achieving certain objectives at the investment horizon while staying within a risk budget. Diversification, hedging, and insurance can be relied on to make optimal use of risk budgets. These three techniques involve different aspects of risk management, but they are complementary techniques rather than competing ones.

Diversification provides investors with the best reward per unit of risk through a smart combination of individual assets. It is designed to work in the long run across different market conditions and is, therefore, helpless in such specific conditions as severe market downturns. Since the purpose of diversification is efficient extraction of risk premia, it is most effective in the construction of performance-seeking portfolios.

Hedging can be combined with diversification to reduce risks that cannot be diversified away. Hedging is achieved through a portfolio of safe assets, or simply through cash, which is another dedicated building block. A non-diversifiable risk that can be handled in this way is the risk of a large drawdown.

Insurance, unlike diversification and hedging, combines the safe building block and the PSP optimally to comply with the corresponding risk budgets. So downside risk control is best achieved through dynamic asset allocation. This technique makes it possible to control the downside of the return distribution while preserving access to the upside through the PSP. In this context, a well-diversified portfolio is a building block of crucial importance. A carefully designed PSP with an improved Sharpe ratio resulting from good diversification can reduce the implicit cost of insurance.

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Appendix
Appendix

Appendix 1: Are GMV Portfolios Generally Concentrated in Low-Volatility Stocks?

If a non-degenerate covariance matrix is assumed, the optimisation problem in Eq. 1 without the expected return constraint makes possible the following analytic solution,

\[ w = \frac{1}{e' \Sigma^{-1} e} \Sigma^{-1} e \]

where \( \Sigma^{-1} \) is the inverse of the covariance matrix. The factor \((e' \Sigma^{-1} e)^{-1}\) in Eq. 7 ensures that the weights add up to one and, therefore, the optimal solution is proportional to the vector \(\Sigma^{-1} e\). This solution is, for all intents and purposes, the GMV portfolio.

Suppose that returns of the assets in the portfolio are not correlated. The inverse of the covariance matrix then has the following very simple structure,

\[ \Sigma^{-1} = \begin{pmatrix} \sigma_{11}^{-1} & 0 & \ldots & 0 \\ 0 & \sigma_{22}^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_{nn}^{-1} \end{pmatrix} \]

where \(\sigma_{jj}\) is the variance of the return of the \(j\)-th asset. In this case, the optimal solution as given in Eq. 7 is proportional to the inverse of the squared volatilities. As a result, the more diverse the volatilities in the universe are, the more highly the GMV portfolio is concentrated in the low-volatility assets.

It is expected that increasing correlations from zero to a positive number will lead to even greater concentration in the low-volatility assets since, as a result of some common factors, it would be optimal to tilt the portfolio further towards the lower-volatility assets. When all assets are perfectly positively correlated, the GMV long-only portfolio is concentrated entirely in the lowest-volatility stock.

A way to illustrate this idea is to consider, for the sake of simplicity, a constant correlation model and different ranges for the stock volatilities. The case in which all stocks have one and the same volatility is not of interest because the GMV portfolios are trivial—equally weighted for any correlation. The expectation is that the more we allow the stock volatilities to vary, the stronger the departure from equal weighting will be.

In this illustration, we choose the inverse of the Herfindahl index as a measure of concentration which, in a vector notation, takes the form,

\[ H^{-1} = \frac{1}{w'w} \]

where \(w\) is a vector of weights (e.g., the optimal solution in Eq. 7). For an equally weighted portfolio, \(H^{-1} = n\), which stands for the number of stocks in the universe, and for a portfolio totally concentrated in one asset, \(H^{-1} = 1\). The concentration metric \(H^{-1}\) is between 1 and \(n\) for any other long-only portfolio.

To preserve this interpretation, we calculate the GMV portfolios using the optimisation problem in Eq. 1 with the additional constraint that the weights should be non-negative. We consider a hypothetical universe of 100 stocks, the annualised volatilities of which are equally spaced in the following ranges: [16%, 18%], [15%, 19%], [13%, 21%], and [10%, 24%]. In the three cases, the average volatility is one and the same; the only difference is the dispersion of the volatilities around the
Appendix

average. On these assumptions, we calculate numerically the GMV portfolios for degrees of correlation ranging from 0 to 0.99 and we estimate the concentration metric $H^{-1}$ for each of them.

Figure 10: The inverse of the Herfindahl index of long-only GMV portfolios as a function of correlation in a constant correlation model. The volatilities are equally spaced between the lower and the upper bounds, which are provided in annualised terms.

The plot of $H^{-1}$ as a function of correlation is provided in figure 10. As expected, the relationship is monotonic in the four cases and, all other things equal, a wider range of volatility corresponds to greater concentration.

The plot in figure 10 indicates that concentration increases, but it is not clear if it does so as a result of the higher relative weight of the low-volatility stocks. To explore this question, we rank the GMV stocks into three groups of equal size and then calculate the sum of the weights of the stocks in each group. Figure 11 shows the sum of these weights as a function of correlation, assuming that the stock volatilities are in the interval [16%, 18%].

If the stock returns are non-correlated, the total weight allocated to the three groups of stocks is almost the same, which is consistent with the value of $H^{-1}$ being near 100 in figure 10. As correlation increases, the total weight allocated to the lower-volatility stocks increases monotonically, an increase that is offset by a monotonic decrease of the total weight allocated to the group of higher-volatility stocks. For correlation more than 0.6, the portfolio is concentrated entirely in the group of lower-volatility stocks.

Figure 11: The sum of the GMV weights corresponding to three groups of stocks of equal size ranked by their volatility. The volatilities of the stocks are equally spaced in the interval [16%, 18%].

This numerical illustration corresponds to the fairly homogeneous case in which stock volatilities do not vary much. The conclusions would be stronger if we allowed a greater degree of non-homogeneity.

Appendix 2: Diversification Opportunities Are Determined by the Multivariate Dependence Structure of Asset Returns

In this appendix, we examine more closely the claim that the functional dependence of asset returns leaves no room for diversification. The analysis is general in the sense that we do not limit the discussion to the mean-variance setting alone. Rather, we work with a general risk measure, denoted by $\rho$ here, making as few assumptions as possible while extending the Markowitz setting.
Suppose, to start, that the investable universe consists of \( n \) assets, the joint behaviour of the returns of which is described by the random vector \((X_1, ..., X_n)\). We distinguish between marginal distributions, describing the elements of the random vector on a standalone basis, and the joint distribution, describing the joint behaviour of all stocks. Let \( \rho \) be a general risk measure that is sub-additive (i.e., it satisfies Eq. 3) and positively homogeneous\(^{26}\) \( \rho(aX) = a\rho(X) \) for positive \( a \); assume that it satisfies the structure neutrality condition (Ekeland et al. 2009),

\[
\text{Eq. 8} \quad \sup_{X_i = x_i} \rho(X_1 + \ldots + X_n) = \rho(X_1) + \ldots + \rho(X_n)
\]

where the supremum is taken with respect to all possible joint distributions; however, the marginal distributions are fixed—that is, we vary only the dependence structure. This condition implies that the dependence structure for which the supremum is attained—call it the worst possible dependence structure—leads to a linear decomposition of the risk measure. If this condition does not hold, then by the sub-additivity property \( \rho(X_1 + \ldots + X_n) < \rho(X_1) + \ldots + \rho(X_n) \) for the worst possible dependence structure. However, the strict inequality means that we must exclude volatility and the multivariate normal distribution as a possible setup. Additional motivation for this condition can be found in Ekeland et al. (2009).

To characterise the worst possible dependence structure, we need the notion of comonotonicity. A random vector \( Y \) is said to be comonotonic if\(^{27}\)

1. The following representation holds

\[
\text{Eq. 9} \quad Y = (y_1, ..., y_n) \sim \left(F_1^{-1}(U_1), ..., F_n^{-1}(U_n)\right)
\]

where \( U \) is uniformly distributed in \([0, 1]\), or

2. The multivariate distribution function of \( Y \) can be represented as

\[
\text{Eq. 10} \quad F_Y(x_1, ..., x_n) = \min(F_{X_1}(x_1), ..., F_{X_n}(x_n))
\]

The definition in Eq. 9 indicates that the components of a comonotonic random vector can be represented as increasing functions of each other. Eq. 10 shows that the upper Fréchet-Hoeffding bound,

\[
\text{Eq. 11} \quad F_X(x_1, ..., x_n) \leq \min(F_{X_1}(x_1), ..., F_{X_n}(x_n))
\]

which is an upper bound of any multivariate distribution function, is the distribution function of a comonotonic random vector. Since the marginal distributions of the upper bound are the same, it follows that comonotonicity is, essentially, a property of the copula, or the dependence structure, of the random vector.

In fact, the worst possible dependence structure turns out to be that of comonotonicity. It is possible to show that for any convex risk measure that satisfies Eq. 8, the following inequality holds\(^{28}\)

\[
\rho(X_1 + \ldots + X_n) \leq \rho(F_{X_1}^{-1}(U) + \ldots + F_{X_n}^{-1}(U)) = \rho(F_{X_1}^{-1}(U)) + \ldots + \rho(F_{X_n}^{-1}(U))
\]

which can be restated in terms of portfolio risk and standalone risks as

\[
\text{Eq. 12} \quad \rho(w_1X_1 + \ldots + w_nX_n) \leq w_1\rho(X_1) + \ldots + w_n\rho(X_n)
\]

That is, holding the marginal distributions fixed, any risk measure satisfying the assumptions above has an absolute
We can conclude that diversification opportunities are determined essentially by the joint behaviour of stock returns. The function of risk measures in this more general setting is to provide an objective to translate those opportunities into actual allocations. If there are no diversification opportunities, however, the choice of objective is irrelevant.

**Appendix 3: Dynamic Portfolio Choice in Continuous Time with an Implicit Lower Bound**

In this appendix, we provide in detail the assumptions and the problem setup leading to the particular solution given in Eq. 5. In terms of assumptions, we do not consider the most general setting.

Consider an economy whose uncertainty is represented through a standard probability space $\Omega$, $A$, $P$ and a finite time span denoted by $T$. Investors trade $n$ assets, the prices of which, represented by the elements of a vector $S_t$, evolve as

$$dS_t = \text{diag}(S_t)(r_t + \sigma_t \lambda_t \text{dt} + \sigma_t \text{dz}_t)$$

in which $z$ is an $n$-dimensional Brownian motion and $\text{diag}(S_t)$ is a diagonal matrix with the elements of $S_t$ on the main diagonal, $r_t$ is a non-stochastic risk-free rate, $\lambda_t$ is the market price of risk vector, and $\sigma_t$ is a time-dependent $n \times n$ matrix. The market is assumed to be complete.

A portfolio strategy is described by a vector process of weights $w_t$ adapted to the filtration of the probability space augmented with the natural filtration of the $n$-dimensional Brownian motion. Letting the value process of the strategy be $A$, we...
obtain the following expression describing its dynamics,

$$dA_t = A_t \left( r_t + w_t^\tau \sigma_t^\tau \lambda_t + w_t^\tau \sigma_t \, dz_t \right).$$

In addition, we assume that there is a benchmark evolving according to the equation

$$dB_t = B_t \left( \mu_B \, dt + \sigma_B \, dz_t \right)$$

and that the floor is a multiple of the benchmark, $F_t = kB_t$. Because we assume that the market is complete, it follows that the benchmark can be replicated by the traded securities.

The investors' utility functions take the following form,

$$V(x) = \begin{cases} 
F(x-k) & x \geq k \\
-\infty & x < k 
\end{cases}$$

where $u(x) = x^{1-\gamma}(1-\gamma)$ is the CRRA utility function and $\gamma$ is the risk-aversion parameter. This definition results in infinite disutility if $x < k$.

Investors maximise expected utility at the terminal time instant $T$ by solving the following expected utility maximisation problem subject to a budget constraint:

$$\max_w \mathbb{E}V \left( \frac{A_T}{B_T} \right)$$

s.t.

$$\mathbb{E}(M_T A_T) = A_0$$

where $A_0$ is the initial capital and $M_T$ the pricing kernel. In addition, there is also a liquidity constraint $A_T \geq 0$ in an almost-sure sense for all $t \leq T$. On these assumptions, the solution to the expected utility maximisation problem in Eq. 13 is given by

$$w_t^* = \frac{1}{\gamma} \left( 1 - \frac{k \bar{B}_t}{A_t^*} \right) w_t^{PSP} + \left( 1 - \frac{1}{\gamma} \left( 1 - \frac{k \bar{B}_t}{A_t^*} \right) \right) w_t^B$$

in which $A_t^*$ is the asset value of the optimal solution and:

Eq. 14 $w_t^{PSP} = (\sigma_t^\tau \sigma_t^*-1) \sigma_t^\tau \lambda_t$

Eq. 15 $w_t^B = (\sigma_t^\tau \sigma_t^*-1) \sigma_t^\tau \sigma_B$

Eq. 16 $d\bar{B}_t = \bar{B}_t \left( r_t + w_t^{PSP} \sigma_t^\tau \lambda_t - w_t^B \sigma_t \, dz_t \right)$

The expression in Eq. 14 stands for the weights of the PSP, Eq. 15 for the weights of the benchmark replicating portfolio, and Eq. 16 defines $\bar{B}_t$ as the stochastic process of the benchmark-replicating portfolio constructed such that $\bar{B}_T = B_T$. 

Appendix
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About EDHEC-Risk Institute
About EDHEC-Risk Institute

The Choice of Asset Allocation and Risk Management
EDHEC-Risk structures all of its research work around asset allocation and risk management. This issue corresponds to a genuine expectation from the market.

On the one hand, the prevailing stock market situation in recent years has shown the limitations of diversification alone as a risk management technique and the usefulness of approaches based on dynamic portfolio allocation.

On the other, the appearance of new asset classes (hedge funds, private equity, real assets), with risk profiles that are very different from those of the traditional investment universe, constitutes a new opportunity and challenge for the implementation of allocation in an asset management or asset-liability management context.

This strategic choice is applied to all of the Institute’s research programmes, whether they involve proposing new methods of strategic allocation, which integrate the alternative class; taking extreme risks into account in portfolio construction; studying the usefulness of derivatives in implementing asset-liability management approaches; or orienting the concept of dynamic “core-satellite” investment management in the framework of absolute return or target-date funds.

An Applied Research Approach
In an attempt to ensure that the research it carries out is truly applicable, EDHEC has implemented a dual validation system for the work of EDHEC-Risk. All research work must be part of a research programme, the relevance and goals of which have been validated from both an academic and a business viewpoint by the Institute’s advisory board. This board is made up of internationally recognised researchers, the Institute’s business partners, and representatives of major international institutional investors. Management of the research programmes respects a rigorous validation process, which guarantees the scientific quality and the operational usefulness of the programmes.

Six research programmes have been conducted by the centre to date:
- Asset allocation and alternative diversification
- Style and performance analysis
- Indices and benchmarking
- Operational risks and performance
- Asset allocation and derivative instruments
- ALM and asset management

These programmes receive the support of a large number of financial companies. The results of the research programmes are disseminated through the EDHEC-Risk locations in London, Nice, and Singapore.
In addition, EDHEC-Risk has developed a close partnership with a small number of sponsors within the framework of research chairs or major research projects:

- **Regulation and Institutional Investment**, *in partnership with AXA Investment Managers*
- **Asset-Liability Management and Institutional Investment Management**, *in partnership with BNP Paribas Investment Partners*
- **Risk and Regulation in the European Fund Management Industry**, *in partnership with CACEIS*
- **Structured Products and Derivative Instruments**, *sponsored by the French Banking Federation (FBF)*
- **Dynamic Allocation Models and New Forms of Target-Date Funds**, *in partnership with UFG-LFP*
- **Advanced Modelling for Alternative Investments**, *in partnership with Newedge Prime Brokerage*
- **Asset-Liability Management Techniques for Sovereign Wealth Fund Management**, *in partnership with Deutsche Bank*
- **Core-Satellite and ETF Investment**, *in partnership with Amundi ETF*
- **The Case for Inflation-Linked Corporate Bonds: Issuers’ and Investors’ Perspectives**, *in partnership with Rothschild & Cie*
- **Advanced Investment Solutions for Liability Hedging for Inflation Risk**, *in partnership with Ontario Teachers’ Pension Plan*
- **Exploring the Commodity Futures Risk Premium: Implications for Asset Allocation and Regulation**, *in partnership with CME Group*
- **Structured Equity Investment Strategies for Long-Term Asian Investors**, *in partnership with Société Générale Corporate & Investment Banking*
- **The Benefits of Volatility Derivatives in Equity Portfolio Management**, *in partnership with Eurex*
- **Solvency II Benchmarks**, *in partnership with Russell Investments*

The philosophy of the Institute is to validate its work by publication in international journals, as well as to make it available through its position papers, published studies, and conferences.

Each year, EDHEC-Risk organises a major international conference for institutional investors and investment management professionals with a view to presenting the results of its research: EDHEC-Risk Institutional Days.

EDHEC also provides professionals with access to its website, [www.edhec-risk.com](http://www.edhec-risk.com), which is entirely devoted to international asset management research. The website, which has more than 42,000 regular visitors, is aimed at professionals who wish to benefit from EDHEC’s analysis and expertise in the area of applied portfolio management research. Its monthly newsletter is distributed to more than 700,000 readers.
About EDHEC-Risk Institute

Research for Business
The Institute’s activities have also given rise to executive education and research service offshoots. EDHEC-Risk’s executive education programmes help investment professionals to upgrade their skills with advanced risk and asset management training across traditional and alternative classes.

The EDHEC-Risk Institute PhD in Finance
www.edhec-risk.com/AIeducation/PhD_Finance
The EDHEC-Risk Institute PhD in Finance is designed for professionals who aspire to higher intellectual levels and aim to redefine the investment banking and asset management industries. It is offered in two tracks: a residential track for high-potential graduate students, who hold part-time positions at EDHEC, and an executive track for practitioners who keep their full-time jobs. Drawing its faculty from the world’s best universities and enjoying the support of the research centre with the greatest impact on the financial industry, the EDHEC-Risk Institute PhD in Finance creates an extraordinary platform for professional development and industry innovation.

FTSE EDHEC-Risk Efficient Indices
www.edhec-risk.com/indexes/efficient
FTSE Group, the award winning global index provider, and EDHEC-Risk Institute launched the first set of FTSE EDHEC-Risk Efficient Indices at the beginning of 2010. Offered for a full global range, including All World, All World ex US, All World ex UK, Developed, Emerging, USA, UK, Eurobloc, Developed Europe, Developed Europe ex UK, Japan, Developed Asia Pacific ex Japan, Asia Pacific, Asia Pacific ex Japan, and Japan, the index series aims to capture equity market returns with an improved risk/reward efficiency compared to cap-weighted indices. The weighting of the portfolio of constituents achieves the highest possible return-to-risk efficiency by maximising the Sharpe ratio (the reward of an investment per unit of risk). These indices provide investors with an enhanced risk-adjusted strategy in comparison to cap-weighted indices, which have been the subject of numerous critiques, both theoretical and practical, over the last few years. The index series is based on all constituent securities in the FTSE All-World Index Series. Constituents are weighted in accordance with EDHEC-Risk’s portfolio optimisation, reflecting their ability to maximise the reward-to-risk ratio for a broad market index. The index series is rebalanced quarterly at the same time as the review of the underlying FTSE All-World Index Series. The performances of the EDHEC-Risk Efficient Indices are published monthly on www.edhec-risk.com.

EDHEC-Risk Alternative Indexes
www.edhec-risk.com/indexes/pure_style
The different hedge fund indexes available on the market are computed from different data, according to diverse fund selection criteria and index construction methods; they unsurprisingly tell very different stories. Challenged by this heterogeneity, investors cannot rely on competing hedge fund indexes to obtain a “true and fair” view of performance and are at a loss when selecting benchmarks. To address this issue, EDHEC Risk was the first to launch composite hedge fund strategy indexes as early as 2003. The thirteen EDHEC-Risk Alternative Indexes are published monthly on www.edhec-risk.com and are freely available to managers and investors.

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For more information, please contact:
Carolyn Essid on +33 493 187 824
or by e-mail to: carolyn.essid@edhec-risk.com

EDHEC-Risk Institute
393 promenade des Anglais
BP 3116
06202 Nice Cedex 3 — France

EDHEC Risk Institute—Europe
10 Fleet Place - Ludgate
London EC4M 7RB - United Kingdom

EDHEC Risk Institute—Asia
1 George Street
#07-02
Singapore 049145

www.edhec-risk.com