Static Mean-Variance Analysis with Uncertain Time Horizon

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Abstract

We generalize Markowitz analysis to the situations involving an uncertain exit time. Our approach preserves the form of the original problem in that an investor minimizes portfolio variance for a given level of the expected return. However, inputs are now given by the generalized expressions for mean and variance-covariance matrix involving moments of the random exit time in addition to the conditional moments of asset returns. While efficient frontiers in the generalized and the standard Markowitz case may coincide under certain conditions, we demonstrate, by means of an example, that in general that is not true. In particular, portfolios efficient in the standard Markowitz sense can be inefficient in the generalized sense and vice versa. As a result, an investor facing an uncertain time-horizon and investing as if her time of exit is certain would in general make sub-optimal portfolio allocation decisions. Numerical simulations show that a significant efficiency loss can be induced by an improper use of standard mean-variance analysis when time-horizon is uncertain.

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There is an interesting discrepancy between finance theory and practice. On the one hand, most of the standard financial economics literature is based on the assumption that, at the moment of making an investment decision, an investor knows with certainty the time of her eventual exit. This assumption can be traced back to the origins of modern financial economics, and, in particular, to the development of portfolio selection theory by Markowitz (1952, 1959). On the other hand, most investors would acknowledge that, upon entering the market, they do not know with certainty when they will exit. The time of exit may or may not depend on the price behavior of one or more assets in the investor’s opportunity set. The examples of the latter are exits from the market as a result of purchasing or selling of a house, loss of a job, early retirement, etc.

Since an investment horizon is practically never known with certainty on the date initial investment decisions are made, it is of both practical and theoretical interest to develop a comprehensive theory of investment for uncertain time horizons. Research on the subject starts with Yaari (1965), who addresses the problem of optimal consumption for an individual with an uncertain date of death, in a simple setup with no uncertainty and a pure deterministic investment environment. Hakansson (1969, 1971) extends this work to a discrete-time setting under uncertainty, including risky assets and an uncertain time-horizon. Merton (1971), as a special case, also addresses a dynamic optimal portfolio selection problem for an investor retiring at an uncertain date, defined as the date of the first jump of a Poisson process with constant intensity. This was done in a continuous-time setting with no bequest motive. In a subsequent work, Richard (1975) generalizes these results to the presence of life insurance, while Blanchet-Scaillet et al. (2002) introduce a bequest motive and randomly time-varying probabilities of exiting the market. Liu and Loewenstein (2002) study an intertemporal portfolio optimization problem with an exponentially distributed time-horizon. Another related paper is Huang (2003), who finds equilibrium liquidity premia in an economy where investors face uncertain exit times.

To the best of our knowledge, none of the existing literature explicitly considers the possibility of an exit time that is dependent on other sources of uncertainty. In this paper we provide the first towards filling this gap: we consider an optimal portfolio selection problem for an investor who maximizes a quadratic expected utility function and faces an uncertain exit time which can be either independent or dependent of asset price behavior. For simplicity, we require that the investor is prevented from rebalancing her portfolio before the time of exit, just as in the standard Markowitz analysis. The form of the original Markowitz problem is preserved in that an investor minimizes portfolio variance, for a given level of the expected return.

However, inputs are now given by the generalized expressions for mean and variance-covariance matrix. These expressions involve moments of the random exit time in addition to the conditional moments of asset returns. Therefore, in determining her optimal investment portfolio, an investor considers both the asset price and the exit time risk. The key question that we address is: does the exit time risk impact the set of efficient portfolios? In other words, should investors worry if they were to simply ignore the exit time risk? When exit time is independent of asset price behavior, the answer to that question is, in rather general circumstances, no. Namely, the set of mean-variance frontier portfolios in that case does not depend on exit time probability distribution, and coincides with the standard (certain exit time) case. This result both confirms and extends the result in Merton (1971) (see the discussion in Section 4). More generally, when the exit time is dependent on portfolio performance the set of mean-variance efficient portfolios may depend on exit time distribution. In particular, we consider a simple exit strategy where conditional asset returns follow a random walk and show that there are portfolios that are inefficient in the standard (certain exit time) Markowitz sense but are efficient in the sense of the generalized Markowitz procedure that takes into account the exit time risk, and vice versa. Therefore, an investor facing an uncertain time-horizon and investing as if her time of exit were certain would in general make sub-optimal portfolio allocation decisions.

The main limitation of our analysis is that it is static by nature. This criticism is not new, and is raised against the standard Markowitz procedure as well. Despite possible criticism, investors around the globe use static mean-variance portfolio analysis when selecting their investment portfolios. Addressing the main question of this paper is, therefore, of significant practical import. From an academic perspective, abstracting away from the
technical complexities of a more general dynamic portfolio problem allows us to relax the assumption of exit time independence when studying optimal portfolio selection. (In truth, we think of this as a first step towards building a fully dynamic portfolio selection theory with dependent exit time).

The rest of the paper is organized as follows. After presenting basic notation and definitions (Section 1), we discuss the properties of returns under time-horizon uncertainty (Section 2). We apply the results of Section 2 to formulate a mean-variance portfolio selection problem under uncertain exit time (Section 3). In Section 4, we address optimal portfolio selection under independent exit time. The main results of the paper can be found in Section 5, where we demonstrate, by means of an example, that generalized and standard Markowitz procedures lead, in general, to different sets of efficient portfolios. Our conclusions and suggestions for further research are presented in Section 6. Some proofs and technical details are found in the appendix.

1. Notation and Definitions

Let us begin by introducing the basic notation and definitions of the standard Markowitz procedure (see, for example, Ingersoll 987, Chapter 4 for more details). We consider $n \geq 2$ linearly independent risky assets traded in a frictionless economy where unlimited short selling is allowed and assume that their rates of return have finite variance-covariance matrix $V$ and the expected returns vector $\mu$. The variance-covariance matrix $V$ is assumed to be symmetric and positive definite.\(^4\)

A portfolio $p$ is called a mean-variance frontier portfolio in the standard Markowitz sense if and only if $w_p$, the vector of weights defining $p$, is the solution to the following quadratic program

$$
\begin{align*}
\min_w \sigma_p^2 & \equiv \min_w w^T \cdot V \cdot w \\
\text{such that} & \\
\mu_p & \equiv E[R_p] = \bar{E} \\
w^T \cdot 1 & = 1
\end{align*}
$$

where $E[R_p] = w^T \cdot \mu$ is the expected rate of return on portfolio $p$, $1$ is a $n$-vector of ones, and $\cdot$ denotes the usual matrix product. Note that the conditions imposed on $V$ are necessary and sufficient for the existence of the unique solution to the optimization problem. Extending the analysis to all possible values of $E[R_p]$, one obtains the mean-variance frontier. Not all mean-variance portfolios are efficient, in that some mean-variance portfolios are dominated. For a portfolio to be efficient, it is necessary, but not sufficient, for it to belong to the mean variance frontier. The subset of the mean-variance frontier which is not dominated is called the efficient frontier, the main output of the Markowitz analysis.

Construction of a mean-variance frontier relies on a number of implicit well-known assumptions, including the assumption that an investor knows exactly when she is going to exit the market: she buys at initial time 0 and sells at time 1.\(^6\) Since it involves only two dates, that is obviously a static problem.

2. Asset Returns under Uncertain Time of Exit

We begin by considering a single risky asset held from date 0 to date $\tau$, where $\tau$ is a positive random variable describing the investor’s time-horizon. We assume that the probability distribution of $\tau$ is known and that it has finite moments of as many orders as necessary for the analysis to make sense.\(^7\) We denote by $G$ cumulative distribution function (CDF) and by $g$ the corresponding probability density function (PDF) of the stochastic exit time.

The return on asset $S$ from date 0 to date $\tau$ is defined as

$$R_{0,\tau} = \frac{S_\tau - S_0}{S_0}$$

---

4 - The vector of expectations $\mu$ must have components not all equal to each other.

5 - Positive-definiteness means that, for an arbitrary $n$-vector of weights $w$ such that $w \neq 0$, we have that $w^T \cdot V \cdot w > 0$. Here, $w^T$ denotes the transpose of the vector $w$, and $w \neq 0$ means that there is at least one element of $w$ which is not equal to zero.

6 - For more details, see the classic treatise by Markowitz (1959).

7 - In the case of the Random Walk (RW) model of asset returns, it is sufficient to assume that the first two moments are finite (see below).
Note that there are two sources of uncertainty here. On the one hand, for a given realization of \( \tau \), the future asset price is uncertain. This sort of risk is called the asset price risk. On the other hand, the realization of \( \tau \) itself is uncertain, which is called the exit time risk. It is intuitively clear that asset allocation techniques in the presence of time-horizon uncertainty should take into account both of these types of risks.

2.1 Dependent versus Independent Time-Horizon

We now make an important distinction between the independent and dependent exit times. Intuitively speaking, an independent exit time is a stochastic exit time that does not depend on the price behavior of any asset in the investor's opportunity set. In order to make this definition more precise, let us denote by \( F_t \) CDF of the conditional distribution of \( R_{0, \tau} \) given \( \tau = t \) and by \( f_t \) the corresponding PDF. Let us, also, denote by \( F \) (respectively, \( f \)) the unconditional CDF (respectively, PDF) of the return process \( R_{0, \tau} \). In that case, by definition

\[
f(r) = \int_{\tau}^{t} f_t(r) g(t) \, dt
\]

where \( \{\tau, t\} \) is the support of \( g \) (with \( \tau \geq t \geq \tau > 0 \)). In technical terms, \( f \) is called a mixture of the density functions \( f_t \) according to \( g \), or equivalently \( f \) is said to be obtained by randomizing the parameter \( t \) in \( f_t \) according to \( g \) (Thomasian 1969). We are now ready to introduce the following definition.

We refer to a positive random variable \( \tau \) as an independent exit time if the conditional distribution of \( R_{0, \tau} \) given \( \tau = t \) is identical to the unconditional distribution of \( R_{0, \tau} \), i.e., if \( f(r) = f_t(r) \). Otherwise, we refer to it as a dependent exit time. Dependent exit time, by definition, depends on the behavior of one or more assets in an investor's opportunity set. Our approach (see equation 7 in section 3) is valid for both dependent and independent exit times. We discuss portfolio selection under independent exit times in Section 4, while an example of dependent exit time is studied in Section 5, namely a stop-loss-like strategy.

Let us consider a very simple independent exit time model: an investor has a probability \( p = 0.5 \) of exiting at time \( \tau = 1 \) and a probability \( 1 - p = 0.5 \) of exiting at time \( \tau = 2 \). If the conditional returns on asset \( S \) have Gaussian distribution with instantaneous mean and volatility equal to \( \mu \) and \( \sigma \), respectively, one obtains the following expression for the (unconditional) density function \( f(r) \)

\[
f(r) = \int_{0}^{\infty} \frac{dt}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}(r-\mu)^2} [p\delta(t-1) + (1-p)\delta(t-2)]
\]

where we have used Dirac's delta function \( \delta(t) \) to capture the fact that the exit is possible only at the two discrete dates. We observe that the density function \( f \) in that case is simply a probability-weighted average of the density functions \( f_1 \) and \( f_2 \).

2.2 First Two Moments of Asset Returns over an Uncertain Time-Horizon

In practice, we may be more interested in moments, rather than in the distribution itself. In mean-variance analysis, in particular, we need only know the first two moments. Using the law of iterated expectations, we obtain that the unconditional expectation of an asset return over the random time horizon \( \tau \) is given by

\[
\mathbb{E}(R_{0, \tau}) = \mathbb{E}[\mathbb{E}(R_{0, \tau} | \tau)]
\]

Thus, the unconditional expected return is a probability-weighted average of the conditional expected returns. This is useful for practical applications since one can typically estimate conditional, rather than unconditional, moments of the portfolio return distribution. Returning to our example with Gaussian returns and a discrete
probability stochastic exit time, we obtain

\[ \mathbb{E}(R_{0,\tau}) = P_{\tau}(\tau = 1) \]

Compare this expression with the two cases: \( \tau = 1 \) and \( \tau = 2 \), with certainty. Expected returns would then be \( \mathbb{E}(R_{0,\tau}) = \mu \) or \( \mathbb{E}(R_{0,\tau}) = 2\mu \), respectively. When exit time is uncertain, therefore, the expected value lies between the two extreme values, which makes intuitive sense.

In exactly the same fashion, one can obtain a decomposition of higher moments. In particular, a decomposition of variance is given by the following expression:

\[ \mathbb{V}(R_{0,\tau}) = \mathbb{E}[\mathbb{V}(R_{0,\tau} | \tau)] + \mathbb{V}[\mathbb{E}(R_{0,\tau} | \tau)] \]

Consider the structure of this expression more closely. It consists of two parts. The first part, \( \mathbb{E}[\mathbb{V}(R_{0,\tau} | \tau)] \), is a probability-weighted average of the conditional variances. It is a measure of the average asset return risk. The second part, \( \mathbb{V}[\mathbb{E}(R_{0,\tau} | \tau)] \), is the variance of the conditional expected returns. It is equal to zero in the case of a certain exit time and can therefore be naturally interpreted as a measure of the exit-time risk. Hence, under a random exit time, risk can be decomposed into the asset return and exit time components.

2.3 The Random Walk Case

We now turn to the general discussion of the random walk case.

**Proposition 1** Let us assume that the return process, conditional on the realization of \( \tau \), follows a random walk with a constant mean \( \mu \) and volatility \( \sigma \). Then, the unconditional mean and variance of the return process over a stochastic time horizon \( \tau \) are given by the following expressions

\[
\mathbb{E}(R_{0,\tau}) = \mu \mathbb{E}(\tau) \\
\mathbb{V}(R_{0,\tau}) = \sigma^2 \mathbb{E}(\tau) + \mu^2 \mathbb{V}(\tau)
\]

where \( \mathbb{E}(\tau) \) and \( \mathbb{V}(\tau) \) respectively are mean and variance of the exit time distribution.

**Proof.** The coefficients \( \mu, \sigma \) are bounded and deterministic functions of time and \( \sigma > 0 \). Since the return process follows a random walk, conditional expected return and variance are linear functions of time horizon. By independence, we obtain

\[
\mathbb{E}(R_{0,\tau} | \tau = t) = \mathbb{E}(R_{0,t}) = \mu t \\
\mathbb{V}(R_{0,\tau} | \tau = t) = \mathbb{V}(R_{0,t}) = \sigma^2 t
\]

The unconditional expected return is equal to

\[ \mathbb{E}(R_{0,\tau}) = \mathbb{E}(\mu t) = \mu \mathbb{E}(\tau) \tag{5} \]

and the variance is

\[ \mathbb{V}(R_{0,\tau}) = \mathbb{E}(\sigma^2 t^2 g(t)) dt + \int \left( \mu t - \mu \mathbb{E}(\tau) \right)^2 g(t) dt \]

\[ = \sigma^2 \mathbb{E}(\tau) + \mu^2 \mathbb{V}(\tau) \tag{6} \]

which concludes the proof.

Again, variance contains two terms: the measure of the asset price risk \( \sigma^2 \mathbb{E}(\tau) \) and the measure of the exit-time risk \( \mu^2 \mathbb{V}(\tau) \). Note also that this is a natural generalization of the fixed exit time case, which corresponds to \( \mathbb{E}(\tau) = T, \mathbb{V}(\tau) = 0 \).
3. Portfolio Selection Problem When Time of Exit is Uncertain

Everything presented so far for a single risky asset generalizes easily to a portfolio of risky assets. In particular, if the return on asset $i$, from date 0 to a random date $\tau$ (assumed to be the same for all assets) is denoted by $R_{0,\tau}^i$, the return on a portfolio $p$, defined by the vector of weights $w$ with components $(w_i)_{i=1,...,n}$, reads

$$R_{0,\tau}^p = \sum_{i=1}^{n} w_i R_{0,\tau}^i$$

The expected return on portfolio $p$ is given by the following expression:

$$\mathbb{E} \left[ R_{0,\tau}^p \right] = \sum_{i=1}^{n} w_i \mathbb{E} \left[ R_{0,\tau}^i \right] = \sum_{i=1}^{n} w_i \mathbb{E} \left[ \mathbb{E} \left( R_{0,\tau}^i \big| \tau \right) \right]$$

Likewise, one can decompose the variance-covariance matrix:

$$\mathbf{V} \left[ R_{0,\tau}^p \right] = \mathbb{E} \left[ \mathbf{V} \left( R_{0,\tau}^p \big| \tau \right) \right] + \mathbb{E} \left[ \mathbb{E} \left( R_{0,\tau}^p \big| \tau \right) \right]$$

Now we are ready to formulate a mean-variance portfolio selection problem under uncertain time of exit. A frontier portfolio is simply a solution $\mathbf{w}_g^\tau$ to the following optimization program

$$\min_{\mathbf{w}} \mathbf{V} \left[ R_{0,\tau}^p \right]$$

such that

$$\mathbb{E} \left[ R_{0,\tau}^p \right] = \overline{E}$$

$$\mathbf{w}^T \cdot \mathbf{1} = 1$$

where $\overline{E}$ is some objective.

When exit time is fixed (Ingersoll 1987, Chapter 4) it is well known that there are two different cases in which mean-variance maximization is consistent with expected utility maximization. The first is when investors's preferences are described by a quadratic utility function (conditional distribution of asset returns can be arbitrary). The second is when preferences are arbitrary but conditional distribution of asset returns belong to a class of elliptic distributions (normal distribution belongs to that class). In this paper we assume that investors have quadratic utility functions and that they are expected utility maximizers. Under the conditions stated in Proposition 2, when exit time is uncertain, the mean-variance maximization problem is consistent with an expected utility maximization problem.

Proposition 2 In case of a quadratic utility function $U \left( \left[ W_{\tau} \right] \right) = W_{\tau} - \frac{1}{2} \left( W_{\tau} \right)^2$, where $W_{\tau}$ is the wealth of an investor at time $\tau$, if there is a positive real number $b$ such that $R_{0,\tau}^p < \frac{1}{b}$ holds almost surely, maximization of the expected utility max $\mathbb{E} U \left( \left[ W_{\tau} \right] \right)$ is equivalent to the mean-variance optimization problem (8) to (10).

The proof mimics the one for the fixed exit time case, and is omitted for brevity.

4 Solution to the Markowitz Problem when Exit Time is Independent

We now demonstrate that when exit time is independent of asset returns, the set of generalized mean-variance frontier weights, i.e., the set of solutions to the generalized Markowitz optimization problems (8) through (10), $\{\mathbf{w}_g^\tau\}$, is identical to the set of frontier weights corresponding to the standard fixed exit time problems (1) to
(3), which we denote as $\{w^*_t\}$ and does not depend on the exit time distribution. As a corollary, we obtain that the mean-variance frontiers as well as the efficient frontiers for the standard and generalized Markowitz optimization problem coincide as long as exit time distribution is independent of asset returns. (To be comparable, mean-variance frontiers have to be plotted in same units, i.e., either in $\tau$ space or in fixed exit time space).

When asset returns follow a random walk and exit time is independent, return on asset $i$ satisfies (see equations (5) (6))

\[ E\left[R^i_{0,\tau}\right] = \mu_i E(\tau) \]
\[ V\left[R^i_{0,\tau}\right] = \sigma^2_i E(\tau) + \mu_i^2 V(\tau) \]

and for a pair of assets $i, j$ decomposition of covariance holds:

\[ C\left[R^i_{0,\tau}, R^j_{0,\tau}\right] = \sigma_i \sigma_j \rho_{ij} E(\tau) + \mu_i \mu_j V(\tau) \]

where $\mu_i, \sigma^2_i$ are, respectively, instantaneous expected value and variance of the return on asset $i$, and $\rho_{ij}$ is the instantaneous correlation coefficient between the two assets (here and below we assume that the exit time is the same for all assets).

Inputs into the generalized Markowitz problem read as follows

\[ E\left[R^p_{0,\tau}\right] = E(\tau) \sum_{i=1}^n \omega_i \mu_i = E(\tau) \mu_p \]
\[ V\left[R^p_{0,\tau}\right] = E(\tau) \sum_{i,j=1}^n \omega_i \omega_j \sigma_{ij} \rho_{ij} + V(\tau) \sum_{i,j=1}^n \omega_i \omega_j \mu_i \mu_j = E(\tau) \sigma^2_p + V(\tau) \mu_p^2 \]

Let us first establish that for any given independent exit time distribution subject to the regularity conditions specified above, if asset returns follow random walk, it is possible to restate the optimization problem (equations (8) through (10)) in terms of a standard portfolio selection program with suitably re-defined parameters. This is the content of the following proposition.

**Proposition 3** If the asset returns follow a random walk and the exit time distribution is independent, the program (8) through (10) is equivalent to the following standard portfolio selection program

\[ \min_{\omega} \omega^T \cdot K \cdot \omega \]

such that

\[ \omega^T \cdot \mu = E^* \]
\[ \omega^T \cdot 1 = 1 \]

where the expected return vector and the variance-covariance matrix are re-defined to account for time-horizon as well as asset price risks.\(^{11}\)

\[ K_{ij} = \sigma_i \sigma_j \rho_{ij} + \frac{V(\tau)}{E(\tau)} \mu_i \mu_j \]
\[ E^* = \frac{E}{E(\tau)} \]

Moreover, a solution to the quadratic programs (13) through (17) exists, and is unique.

**Proof.** See Appendix A.

Thus, when exit time is independent, the portfolio selection problem under uncertain exit time can be reformulated in terms of the standard quadratic mean-variance optimization with adjusted values of the parameters. In addition, the solution to the problem exists and is unique. Now we explicitly solve the problem, and show that the solution is actually identical to the standard Markowitz solution in a certain exit-time case.

\(^{11}\) In the formula the quantity $\frac{E}{E(\tau)}$ could be regarded as an exit-time analogue of a Sharpe ratio. The difference between $K$ and the standard value $\sigma_{ij} \rho_{ij}$ is proportional to that quantity, and, hence, vanishes if and only if this "Sharpe ratio" does, thus re-enforcing the interpretation of that term as a measure of exit time risk.
Proposition 4 If exit time $\tau$ does not depend on the portfolio performance and asset prices follow a random walk, the unique solution to the quadratic programs (13) through (17) is given by

$$w^*_g = \frac{C}{AC} E^* - \frac{B}{B^2} V^{-1} \cdot \mu + \frac{A}{AC} \cdot \frac{B E^*}{B^2} V^{-1} \cdot 1$$

with

$$A = \mu^T \cdot V^{-1} \cdot \mu$$
$$B = \mu^T \cdot V^{-1} \cdot 1$$
$$C = 1^T \cdot V^{-1} \cdot 1$$

and

$$e^* \equiv \frac{E}{E(\tau)}$$

Proof. See appendix B where we demonstrate that this result can be generalized to a much wider class of asset return processes than the random walk as long as exit time is independent.

In other words, even though matrix $K$ depends on $\frac{\tau}{E(\tau)}$, the solution given in equations (18) to (19) does not depend on $\frac{\tau}{E(\tau)}$ and, therefore, on the exit time distribution. Thus, when exit time is independent, the corresponding optimal portfolio weights are independent of the exit time distribution. Furthermore, the form of the solution is exactly the same as the solution in the standard case (to make the comparison easier, we have used a standard notation adopted, for example, in Huang and Litzenberger 1988).12

This result can be seen as i) a confirmation and ii) an extension of a result derived by Merton (1971) and Richard (1975) in a dynamic setup. That our result confirms the one derived in Merton (1971) in the case of the random walk is not surprising. Namely, when exit time is independent of the portfolio composition, an investor's utility function belongs to the class of HARA utilities (quadratic utility is a member of that class), and the risky asset prices follow Geometric Brownian Motion, Merton (1971) demonstrates that the optimal portfolio consumption problem with continuous rebalancing maps into the infinite-horizon problem with a re-normalized subjective discount factor. As a result, the optimal portfolio allocation policy in that case is myopic. On the other hand, and this is a new result, because of the tractability of the static approach we are able to extend Merton's results by showing (in Appendix B) that this invariance result is valid even if one relaxes the assumption of zero serial correlation in conditional asset returns.13

We now provide an even stronger departure from the existing literature by relaxing the assumption of an independent time-horizon and turning to a specific example of a dependent exit time.

5. Dependent Exit Time

In the previous section we have seen that when exit time does not depend on portfolio composition, the set of frontier portfolios is the same as in the certain exit time case. If exit time depends on portfolio composition, proposition 4 is no longer valid. In order to prove that the set of efficient portfolios does not coincide, in general, for investors with fixed and random time horizons, consider the following simple example. Suppose that an investor can exit either at time 1 or at time 2. Further, assume that conditional asset returns in the economy are independent and identically distributed normal variables. Finally, assume that one exits early (at time 1) if and only if portfolio return at that time falls below a certain threshold value $\varepsilon$; otherwise one exits at time 2. A possible motivation for this kind of an exit-time model can be found in the hedge fund universe, where the presence of high-water mark provisions provides incentives to the managers to terminate their fund in the event of a large drawdowns (Goetzmann, Ingersoll, Ross 2003).

12 - It should be noted that this result holds provided that one substitutes the set of all possible values for $\frac{\tau}{E(\tau)}$ for the set of all possible values for $E$. If one normalizes $\frac{\tau}{E(\tau)}$ to be equal to one, then obviously $\frac{\tau}{E} = \frac{\tau}{E(\tau)}$.

13 - Merton (1971) and Richard (1975) assume that asset returns are driven by a geometric Brownian motion, hence ruling out a possible deviation from the random walk.
It is easy to verify that the probability of an early exit is given by $p = \mathcal{N}\left(\frac{\bar{\mu} - \bar{\sigma}}{\bar{\sigma}}\right)$ (here and below, $\mathcal{N}(.)$ and $n(.)$ stand for the cumulative distribution function and density function of the standard normal distribution respectively; also, $\mu_p, \sigma_p$ are defined in (1) and (3), respectively). The following proposition states the optimization problem.

**Proposition 5** The optimization problem corresponding to the above dependent exit time specification reads as follows

$$
\min_w \mathcal{V}(R_{0,T}^p) = \min_w \left( p (1 - p) \mu_p^2 + (2 - p) \sigma_p^2 + 2 \mu_p \sigma_p \sqrt{\frac{1}{2\pi} \frac{1}{\sigma_p^2} e^{-\frac{(\bar{\mu} - \bar{\sigma})^2}{2\sigma_p^2}}} \right) \tag{20}
$$

$$
\mathbb{E}(R_{0,T}^p) = \mu_p (2 - p) - \bar{\mu}, \hspace{1cm} w^T \cdot 1 - 1 \tag{21}
$$

where $p = \mathcal{N}\left(\frac{\bar{\mu} - \bar{\sigma}}{\bar{\sigma}}\right)$.

**Proof.** See Appendix C.

It is easy to verify that $\mathbb{E}(\tau) = 2 - p$, and $\mathcal{V}(\tau) = p(1 - p)$. Thus, in comparison with the independent exit time case (c.f. equations (11) and (12)) there is an extra non-quadratic term in the expression for variance $(2\mu_p \sigma_p \sqrt{\frac{1}{2\pi} \frac{1}{\sigma_p^2} e^{-\frac{(\bar{\mu} - \bar{\sigma})^2}{2\sigma_p^2}}})$; in addition, the probability of an early exit, $p$, is a (non-linear) function of portfolio weights.

Let us now compare the set of solutions to the generalized portfolio optimization problem (20) to (21), with the set of solutions to the standard Markowitz portfolio optimization problem (1) to (3), $\{w^*_g\}$. Even though the corresponding Lagrangians are obviously quite different from each other, in certain conditions the two sets of solutions coincide, i.e., $\{w^*_g\} = \{w^*_s\}$. This happens whenever the parameter $\varepsilon$ is sufficiently large in absolute value. In that case, the third term in (21) becomes negligibly small. If $\varepsilon$ is a large positive number, then an early exit becomes inevitable, i.e., $p \to 1$, in which case the problem is effectively equivalent to the fixed exit time problem with $T = 1$. On the other hand, if $\varepsilon$ is a sufficiently large (in absolute value) and negative, $\varepsilon \to 0$ and no early exit can occur, in which case the problem is effectively equivalent to the fixed exit time problem with $T = 2$. Based on the results of proposition 4, in both of these limiting cases $\{w^*_g\} = \{w^*_s\}$.

Consider now intermediate values of $|\varepsilon|$, which will be the focus of our attention in the numerical simulations below. In that case the term $(2\mu_p \sigma_p \sqrt{\frac{1}{2\pi} \frac{1}{\sigma_p^2} e^{-\frac{(\bar{\mu} - \bar{\sigma})^2}{2\sigma_p^2}}})$ may have a non-negligible influence on the results of the optimization. It is in that region that we expect to find cases for which $\{w^*_g\} \neq \{w^*_s\}$. In particular, we are interested in false positive and false negative types of discrepancies between the two sets of portfolios. A false positive is a portfolio (defined by its set of weights $w$) that is on the standard Markowitz efficient frontier, but is not on the generalized Markowitz efficient frontier. A false negative is a portfolio (defined by its set of weights $w$) that is on the generalized Markowitz efficient frontier, but not on the standard Markowitz efficient frontier. One legitimate question is whether the deviation from the standard situation induced by the presence of a random time-horizon is significant. After all, if the presence of a random time-horizon has only a marginal effect, a fund manager would not have sufficient incentive to revise the asset allocation model he or she has been using. As is transparent from the previous discussion and from the inspection of the optimization problem (20) to (21), we obtain in particular that the effect of the presence of the additional term is i) smaller (larger) when $|\varepsilon|$ is larger (smaller) and ii) larger (smaller) when the portfolio expected return $\mu_p$ is larger (smaller).

In order to understand circumstances in which large deviations between generalized and standard Markowitz efficient frontier can occur, we generate a stylized economy including four assets with identical pairwise correlation coefficients (equal to 0.4), identical volatilities (equal to 25%), and identical expected returns $\mu$ (which we vary along with $\varepsilon$ in the process of the experiment). Since no closed-form solutions are available for the problem (20) to (21), we perform a numerical experiment which involves the five following steps. Step 1: For a fixed $\mu$, find the standard Markowitz global minimum variance portfolio. Step 2: Then, for different $\varepsilon$ values, estimate the corresponding expected return and variance over the random time-horizon. Step 3: For each $\varepsilon$ value, find the generalized minimum variance portfolio with a target expected return equal to the one found
in step 2. If the portfolio obtained in step 3 does not coincide with the portfolio obtained in step 1, we have found an example of a false positive. Step 4: estimate the distance between the two portfolios (procedure shall be discussed in the next paragraph). Step 5: Repeat the procedure for various values of $\mu$ and $\epsilon$.

There are several ways in which one can measure distance between portfolios. One first natural way is to measure the difference in portfolio allocation, using the so-called Manhattan distance formally defined as

$$d(p, p') = \frac{1}{4} \sum_{i=1}^{4} |w_{ip'} - w_{ip}|$$

where $w_{ip'}$ is the allocation to asset $i$ in portfolio $p'$, and where $w_{ip}$ is the allocation to asset $i$ in portfolio $p$.

Another natural definition of distance is the ratio of volatility estimates between the standard and comparable generalized minimum variance portfolios. Finally, the distance can also be measured through certainty equivalence loss. This quantity is determined as an additional investment that has to be made into the inefficient (i.e., standard Markowitz) minimum variance portfolio to reach the same level of utility as using the efficient (i.e., generalized Markowitz) minimum variance portfolio for the same level of expected return. In what follows, we consider a simple quadratic utility function consistent with the mean-variance setup, defined as

$$U(R) = E(R) - \frac{1}{2} \lambda \text{Var}(R)$$

where $R$ is the return over one year (fixed time horizon) and $\lambda$ is a constant for which we consider two different values (5 and 10).

Figure 1 shows the impact of changes in $\mu$ and $\epsilon$ on portfolio distance expressed in terms of portfolio weights. This graph confirms the intuition that the average difference in portfolio weights (in absolute terms) increases in $\mu$ and decreases in the absolute value of $\epsilon$, ceteris paribus. Also, for reasonable values of annual expected returns (say, 20%), portfolios have significantly different portfolio composition.

Figure 2 plots the portfolio distance in terms of volatility ratios. As it can be seen, the volatility of the (comparable) minimum variance portfolio under the assumption of a random time-horizon is always smaller than that of the standard minimum variance portfolio, indicating that an efficiency loss is induced by an improper use of standard mean-variance analysis when time-horizon is uncertain. As expected, the distance is largest when $\epsilon$ is small in absolute terms and as $\mu$ values are increasing. This figure suggests that the efficiency loss can be rather significant, very often greater than 50%.

6. Conclusion and Possible Extensions

Uncertainty over exit time is an important practical issue facing most, if not all, investors. On the theoretical level, this question has not received much attention in the literature, and we have attempted to rectify the situation. We consider the problem of an optimal investment portfolio for an investor who follows a buy-and-hold strategy, but does not know with certainty when she is going to exit and liquidate the portfolio. In order to deal with such problems, we generalize Markowitz analysis to the situations involving uncertainty over the time of exit. Our procedure, while simple, provides an important insight. Namely, it shows that an investor whose time of exit is uncertain is facing two types of risks: an asset price risk and an exit time risk. In order to select an optimal portfolio both of these risks must be addressed. Standard, or fixed exit time, Markowitz procedure is a special case of the generalized procedure developed in this paper.

We show that, when the time of exit does not depend on the portfolio performance, mean-variance frontier portfolios in the generalized sense are identical to the mean-variance frontier portfolios in the standard, fixed exit time case. Furthermore, the set of efficient portfolios does not depend on the exit time distribution (for a large class of exit time and conditional return distributions). On the other hand, when the exit time depends on the performance of the portfolio, the optimization program is much more complex. In this case, the set of optimal portfolios, in general, depends on the details of the exit time distribution and the standard Markowitz portfolio selection method leads, in general, to sub-optimal portfolio allocations. In order to achieve mean-
variance efficiency, therefore, an investor facing an uncertain time of exit should use the approach developed in this paper.

The main limitation of our analysis is that it is static by nature. In the future it would be interesting to explore a fully dynamic portfolio selection model with dependent exit time, provided that the problem can be cast in a tractable setup. It would also be interesting to explore the equilibrium implications of an uncertain time-horizon, along the line of the Huang (2003) analysis of the liquidity premium.
Figure 1: Average difference in portfolio weights. This figure represents the estimate of the distance between the standard global minimum variance portfolio and the corresponding efficient portfolio in a generalized sense, that is when accounting for the presence of an uncertain time horizon, as a function of $\epsilon$ and $\mu$. This is expressed in terms of the Manhattan distance, that is the average difference between the allocation to each asset between the standard efficient portfolio and the corresponding efficient portfolio in a generalized sense. A value equal to 10% indicates that allocation to asset $j$ in the standard and the comparable generalized portfolio can deviate one from another by 10% (in absolute values); an example would be (25%, 25%, 25%, 25%) versus (15%, 15%, 35%, 35%).

Figure 2: Measure of inefficiency of standard efficient portfolios. This figure represents the estimate of the distance between the standard global minimum variance portfolio and the corresponding efficient portfolio in a generalized sense, that is when accounting for the presence of an uncertain time horizon, as a function of epsilon and expected returns ($\mu$). This distance is expressed in terms of ratio of volatilities between the standard efficient portfolio and the corresponding efficient portfolio in a generalized sense. A value equal to 50% indicates that the volatility (over a random time-horizon) of the standard efficient portfolio is twice that of the corresponding efficient portfolio in a generalized sense.
Figure 3: Certainty equivalent monetary loss - case $\lambda = 5$. This figure represents the estimate of the distance between the standard global minimum variance portfolio and the corresponding efficient portfolio in a generalized sense, that is when accounting for the presence of an uncertain-time-horizon, as a function of $\epsilon$ and $\mu$. This distance is expressed in terms of a certainty equivalent monetary loss. A value equal to 10% indicates an investor should invest 110 in the standard efficient minimum variance portfolio to achieve the same utility as what is obtained from investing 100 in the (comparable) efficient portfolio in a generalized sense.

Finally, we display in figure 3 (figure 4) the distance in terms of certainty equivalent monetary loss in case the parameter value is $\lambda = 5$ ($\lambda = 10$).

We can see that the largest monetary loss is obtained for large values of $\mu$ and small absolute values of $\epsilon$. We also obtain that the “cost” of implementing inefficient portfolio strategies when failing to take into account the presence of an uncertain time-horizon can be quite substantial for reasonable parameter values. Our main result is therefore that portfolios efficient in the standard sense can be inefficient in the generalized sense and vice versa. This analysis of false positive portfolios suggests that investors with an uncertain time horizon would in general invest sub-optimally if they were to ignore the exit time risk by using the standard Markowitz method. The results presented here also suggest that inefficiencies induced by not taking into account the presence of a random time-horizon can be significant. In a similar fashion, one could analyze false negative cases.

In conclusion, because of the simplicity and tractability of the static approach, we are able to show that uncertainty over time horizon affects the portfolio decision, provided that one relaxes the assumption of a time horizon (independent of asset returns). This result stands in sharp contrast to the results of Merton (1971), who discusses only the independent exit time case.
Figure 4: Certainy equivalent monetary loss - case $\lambda = 10$. This figure represents the estimate of the distance between the standard global minimum variance portfolio and the corresponding efficient portfolio in a generalized sense, that is when accounting for the presence of an uncertain time horizon, as a function of epsilon and expected returns ($\mu$). This distance is expressed in terms of a certainty equivalent monetary loss. A value equal to 10% indicates an investor should invest 110 in the standard efficient minimum variance portfolio to achieve the same utility as that obtained from investing 100 in the (comparable) efficient portfolio in a generalized sense.
7. References


A. Proof of Proposition 3
We commence by constructing the Lagrangian function corresponding to the maximization problem ((10) through (11)):

\[ L = \frac{\mathbb{E}(\tau) \cdot w}{2} \cdot V \cdot w + \frac{\mathcal{V}(\tau)}{2} (w^T \cdot \mu)^2 + \lambda_3 (\mathbb{E} - \mathbb{E}(\tau) w^T \cdot \mu) + \lambda_2 \mathbb{E}(\tau) (1 - w^T \cdot 1) \] (22)

The first order condition (F.O.C.) for (22) with respect to the portfolio weights reads

\[ \mathbb{E}(\tau) V \cdot w + \mathcal{V}(\tau) (w^T \cdot \mu) \cdot \mu - \lambda_1 \mathbb{E}(\tau) \mu - \lambda_2 \mathbb{E}(\tau) \cdot 1 = 0 \] (23)

while F.O.C. with respect to the Lagrange multipliers yields

\[ \mathbb{E}(\tau) w^T \cdot \mu = \mathbb{E} \]
\[ w^T \cdot 1 = 1 \] (24)

Equations (23) to (24) can be re-written as

\[ V \cdot w + \frac{\mathcal{V}(\tau)}{\mathbb{E}(\tau)} (w^T \cdot \mu) \cdot \mu - \lambda_1 \mu - \lambda_2 \cdot 1 = 0 \] (25)

and

\[ w^T \cdot \mu = \frac{\mathbb{E}}{\mathbb{E}(\tau)} \]
\[ w^T \cdot 1 = 1 \] (26)

One is easily convinced that the equations (25) to (26) are the same as the F.O.C. to a problem associated with the following Lagrangian

\[ L' = \frac{1}{2} w^T \cdot V \cdot w + \frac{\mathcal{V}(\tau)}{2 \mathbb{E}(\tau)} (w^T \cdot \mu)^2 + \lambda_1 \left( \frac{\mathbb{E}}{\mathbb{E}(\tau)} - w^T \cdot \mu \right) + \lambda_2 \left( 1 - w^T \cdot 1 \right) \]
\[ = \frac{1}{2} w^T \cdot K' \cdot w + \lambda_1 \left( \mathbb{E}^* - w^T \cdot \mu \right) + \lambda_2 \left( 1 - w^T \cdot 1 \right) \] (27)

where, in the second line, we have used (16) and (17). Varying values of \( \mathbb{E} \) spans the mean-variance frontier in the same way as \( \mathbb{E}^* \). To prove the second part of the proposition, it is sufficient to prove that \((K'_{ij})_{i,j=1,...,n}\) is a variance-covariance matrix, i.e., that it is symmetric and positive-definite (non-singularity then follows from the Sylvester's Theorem (Baker 2001). Since \( \left( \frac{\mathcal{V}(\tau)}{\mathbb{E}(\tau)} \right) \) is clearly symmetric, and \((V'_{ij})_{i,j=1,...,n}\) is symmetric by assumption, then \((K'_{ij})_{i,j=1,...,n}\) is symmetric as well. Finally, to prove positive definiteness note that, for any vector \( w \neq 0 \), \( w^T \cdot \mathbb{E} \cdot w > 0 \), since \( V \) is positive-definite by assumption.

Therefore

\[ w^T \cdot K' \cdot w = w^T \cdot V \cdot w + \frac{\mathcal{V}(\tau)}{\mathbb{E}(\tau)} (w^T \cdot \mu)^2 \geq w^T \cdot V \cdot w > 0 \]

since \( \mathbb{E}(\tau) \) and \( \mathcal{V}(\tau) \) are both strictly positive, and \((w^T \cdot \mu)^2\) is obviously non-negative. Thus, \( K \) satisfies sufficient conditions for the existence of a unique minimum.

B. Efficient Portfolios for Return Processes More General than Random Walk
Before we prove a generalized version of proposition 4, we relax the assumptions on asset returns by assuming that the conditional variance-covariance matrix is proportional to some positive function \( h \) of time horizon, while the expected return is kept as a linear function of time horizon. In particular, for a single risky asset:
Suppose, for example, that \( h(t) = t^{2H} \), where \( H \) is known as the Hurst exponent equal to 1/2 in the random walk case. In that case, \( H \neq 1/2 \) can be regarded as a measure of serial correlation in conditional asset returns and, thus, a deviation from a random walk (Peters 1991).

Then, using same notations as in the body of the text, unconditional expected return is equal to

\[
\mathbb{E}(R_{0,T}) = \mathbb{E} \left[ \mathbb{E}(R_{0,T} | \tau) \right] = \int_{t=a}^{b} \mu(t) g(t) dt = \mu \mathbb{E}(\tau)
\]

and the variance is

\[
\mathbb{V}(R_{0,T}) = \mathbb{E} \left[ \mathbb{E}(R_{0,T} | \tau) \right] + \mathbb{V} \left[ \mathbb{E}(R_{0,T} | \tau) \right] = \sigma^2 \int_{t=a}^{b} h(t) g(t) dt + \int_{t=a}^{b} (\mu(t) - \mu \mathbb{E}(\tau))^2 g(t) dt
\]

\[
= \sigma^2 \int_{t=a}^{b} h(t) g(t) dt + \mu^2 \int_{t=a}^{b} \left( t - \mathbb{E}(\tau) \right)^2 g(t) dt
\]

\[
= \sigma^2 \mathbb{E}(h(\tau)) + \mu^2 \mathbb{V}(\tau)
\]

The Lagrangian now reads

\[
L = \frac{1}{2} \mathbb{E}(h(\tau)) w^T \cdot V \cdot w + \frac{\mathbb{V}(\tau)}{2} \left( w^T \cdot \mu \right)^2 + \lambda_1 \left( \mathbb{E}(\tau) - w^T \cdot \mu \right) + \lambda_2 \mathbb{E}(h(\tau)) \left( 1 - w^T \cdot 1 \right)
\]

If \( \mathbb{E}(h(\tau)) \) and \( \mathbb{V}(\tau) \) do not depend on the portfolio weights \( w \), the unique solution to the quadratic program (8) through (10) is

\[
w^* = \frac{CE^* + B}{AC - B^2} V^{-1} \cdot \mu + \frac{A - BE^*}{AC - B^2} V^{-1} \cdot 1
\]

with

\[
A \equiv \mu^T \cdot V^{-1} \cdot \mu
\]

\[
B \equiv \mu^T \cdot V^{-1} \cdot 1
\]

\[
C \equiv 1^T \cdot V^{-1} \cdot 1
\]

and

\[
E^* \equiv \frac{\overline{E}}{\mathbb{E}(\tau)}
\]

14 - Here, we assume that all variances and covariances scale identically.
Proof. The Lagrangian for the problem reads

\[ L = \frac{1}{2} \mathbb{E} (h(\tau)) w^T \cdot V \cdot w + \frac{\mathbb{V}(\tau)}{2} (w^T \cdot \mu)^2 + \lambda_1 (\mathbb{E}(\tau) w^T \cdot \mu) + \lambda_2 \mathbb{E} (h(\tau)) (1 - w^T \cdot 1) \]

The following are first-order conditions with respect to \( w \)

\[ \mathbb{E} (h(\tau)) V \cdot w^* + \mathbb{V}(\tau) (w^*^T \cdot \mu) \mu - \lambda_1 \mathbb{E}(\tau) \mu - \lambda_2 \mathbb{E} (h(\tau)) \cdot 1 = 0 \]

while the first order conditions with respect to the Lagrangian multipliers, respectively, read

\[ \overline{E} = \mathbb{E}(\tau) w^*^T \cdot \mu \]

and

\[ 1 = w^*^T \cdot 1 \]

Substituting the first constraint into the objective function yields

\[ \mathbb{E} (h(\tau)) V \cdot w^* + \mathbb{V}(\tau) \frac{\overline{E}}{\mathbb{E}(\tau) \mathbb{E} (h(\tau))} \mu - \lambda_1 \mathbb{E}(\tau) \mu - \lambda_2 \mathbb{E} (h(\tau)) \cdot 1 = 0 \]

It follows that

\[ V \cdot w^* - \mu (\lambda_1 b - a) - \lambda_2 \cdot 1 = 0 \]

where we have defined

\[ a \equiv \frac{\mathbb{V}(\tau) \overline{E}}{\mathbb{E}(\tau) \mathbb{E} (h(\tau))} \]

\[ b \equiv \frac{\mathbb{E}(\tau)}{\mathbb{E} (h(\tau))} \]

The solution takes the following form

\[ w^* = (b \lambda_1 - a) V^{-1} \cdot \mu + \lambda_2 V^{-1} \cdot 1 \]

Next we use the constraint to derive an expression for \( \lambda_1 \) and \( \lambda_2 \)

\[ \left\{ \begin{array}{l}
\mu^T \cdot w^* = (b \lambda_1 - a) \mu^T \cdot V^{-1} \cdot \mu + \lambda_3 \mu^T \cdot V^{-1} \cdot 1 = \frac{\overline{E}}{\mathbb{E}(\tau)} \equiv E^* \\
1^T \cdot w^* = (b \lambda_1 - a) 1^T \cdot V^{-1} \cdot \mu + \lambda_2 1^T \cdot V^{-1} \cdot 1 = 1
\end{array} \right. \]

Defining the following (scalar) quantities

\[ A \equiv \mu^T \cdot V^{-1} \cdot \mu \]

\[ B \equiv \mu^T \cdot V^{-1} \cdot 1 \]

\[ C \equiv 1^T \cdot V^{-1} \cdot 1 \]

we obtain

\[ \left\{ \begin{array}{l}
E^* = A (b \lambda_1 - a) + B \lambda_2 \\
1 = B (b \lambda_1 - a) + C \lambda_2
\end{array} \right. \]
or equivalently

\[
\begin{align*}
Ab\lambda_1 + B\lambda_2 &= aA + E^* \\
Bb\lambda_1 + C\lambda_2 &= aB + 1
\end{align*}
\]

The solution to this system of equations reads

\[
\lambda_1 = \frac{C(aA + E^*) - B(ab + 1)}{AbC - B^2b} = \frac{CaA + CE^* - aB^2 - B}{AbC - B^2b} = \frac{CE^* - B}{(AC - B^2)b} + \frac{a}{b}
\]

\[
\lambda_2 = \frac{Ab(ab + 1) - Bb(aA + E^*)}{(AC - B^2)b} = \frac{A - BE^*}{AC - B^2}
\]

Plugging these expressions back into the expression for \(w^*\) we obtain

\[
w^* = \frac{CE^* - B}{AC - B^2V^{-1}} \cdot \mu + \frac{A - BE^*}{AC - B^2V^{-1}} \cdot 1
\]

which concludes the proof of the proposition.

C. Dependent Exit Time Example - Calculating Moments of Return Distribution

Let us denote by \(R_1\) (\(R_2\)) the return at the one month horizon (two month horizon) and \(f_1\) (\(f_2\)) the associated density function. Under the assumption that asset returns follow random walk, \(R_1 \sim \mathcal{N}(\mu_p, \sigma_p)\) and \(R_2 \sim \mathcal{N}(2\mu_p, \sqrt{2}\sigma_p)\). From the Law of iterated expectations

\[
E(R_{2|1}) = p \times E[R_1 | R_1 < \varepsilon] + (1 - p) \times E[R_3 | R_1 > \varepsilon]
\]

Since \(R_2 = R_1 + R\), where \(R\) has the same distribution as \(R_1\) and is independent from \(R_1\), it follows that

\[
E[R_2 | R_1 > \varepsilon] = E[R_1 | R_1 > \varepsilon] + E[R | R_1 > \varepsilon]
\]

The second term in this expression is equal to \(\mu_p\). On the other hand,

\[
p \times E[R_1 | R_1 < \varepsilon] + (1 - p) \times E[R_3 | R_1 > \varepsilon] = E[R_1] = \mu_p
\]

Thus, we obtain

\[
E(R_{2|1}) = p \times E[R_1 | R_1 < \varepsilon] + (1 - p) \times E[R_2 | R_1 > \varepsilon] = \mu_p (1 + 1 - p) = \mu_p (2 - p)
\]

Now we calculate the expression for the portfolio variance. First, we have that

\[
E \left[ \left( R^e_{2|1} \right)^2 \right] = p \times E \left[ R^e_1 | R_1 < \varepsilon \right] + (1 - p) \times E \left[ R^2_1 | R_1 > \varepsilon \right] = p \times E \left[ R^e_1 | R_1 < \varepsilon \right] + (1 - p) \times E \left[ R^2_1 + R^2 + 2R_1R_2 | R_1 > \varepsilon \right]
\]

\[
= E \left[ R^e_1 \right] + (1 - p) E \left[ R^2 \right] + 2(1 - p) \mu_p E \left[ R_1 | R_1 > \varepsilon \right] \mu_p
\]

\[
= (\sigma_p^2 + \sigma^2_e) (2 - p) + 2(1 - p) \mu_p \mu_\sigma \int_{\varepsilon}^{\infty} \exp \left( -\frac{(x - \mu_p)^2}{2\sigma^2_\sigma} \right) dx
\]

Since

\[
p \times [E(R_1) | R_1 > \varepsilon] = \sqrt{\frac{1}{2\pi\sigma^2_\sigma}} \int_{\varepsilon}^{\infty} x e^{-\frac{(x - \mu_p)^2}{2\sigma^2_\sigma}} dx
\]
using the change of variable \( u = \frac{(x - \mu_p)}{2\sigma_p} \) and the appropriate change in the limits of integration, it follows that

\[
p \times \left[ \mathbb{E}(R_t) \right]_{R_t > \varepsilon} = \mu_p N \left( \frac{\mu_p - \varepsilon}{\sigma_p} \right) + \sqrt{\frac{1}{2\pi}} \int_{\varepsilon/\sigma_p}^{\infty} e^{-u^2} du
\]

\[
= \mu_p (1 - p) + \sigma_p^2 \sqrt{\frac{1}{2\pi\sigma_p^2}} e^{-\frac{(x - \mu_p)^2}{2\sigma_p^2}}
\]

This implies that

\[
\mathbb{E} \left[ \left( R_{0,t}^p \right)^2 \right] = (\mu_p^2 + \sigma_p^2) (2 - p) + 2\mu_p^2 (1 - p) + 2\mu_p \sigma_p \sqrt{\frac{1}{2\pi}} e^{-\frac{(x - \mu_p)^2}{2\sigma_p^2}}
\]

Finally, the variance reads

\[
\mathbb{V} \left( R_{0,t}^p \right) = \mathbb{E} \left[ \left( R_{0,t}^p \right)^2 \right] - \left[ \mathbb{E} \left( R_{0,t}^p \right) \right]^2 = p (1 - p) \mu_p^2 + (2 - p) \sigma_p^2 + 2\mu_p \sigma_p \sqrt{\frac{1}{2\pi}} e^{-\frac{(x - \mu_p)^2}{2\sigma_p^2}}
\]