Is there a gain to explicitly modelling extremes?
A risk management analysis

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Abstract
There is a requirement for decisions taken by risk managers in investment banks to be based upon reliable measures. Most of the time some strong assumptions are made to simplify the estimation process and there has to be a trade-off between ease of estimate and accuracy. In this paper we develop a copula-based approach in order to estimate the Value-at-Risk of portfolios containing financial assets. We propose a survival copula that could solve many difficulties that risk managers currently have to face. We compare the results it provides to those of more classic copulas, on portfolios composed of two and three indices between 1991 and 2005, so that our study covers various market trends. The Heavy Right Tail copula we propose fulfills the various backtest constraints required by regulators.

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EDHEC pursues an active research policy in the field of finance. The EDHEC Risk and Asset Management Research Centre carries out numerous research programmes in the areas of asset allocation and risk management in both the traditional and alternative investment universes.
1. Introduction
Risk management constitutes a core element which conditions the activities of investment banks. To keep banks from encountering any catastrophic situations and to limit the possible propagation of systemic risk, several methods and tools have been developed. Since 1993, Value-at-Risk (VaR) has progressively become a new standard. The success of the concept is probably due to the fact that the measure can be easily understood by all the stakeholders of investment banks. In fact, VaR corresponds to the possible level of loss that can be sustained for a given period and for a certain confidence level. So VaR corresponds to a quantile of the profit and loss (P&L) distribution. Even though the concept is very simple, its computation is more problematic. Several approaches have been developed, but none can claim to be problem-free. The historical approach assumes that the probability distribution of the risk factors is stable over time. Of course, this does not allow losses that have not already been observed over the period of analysis to be obtained. As an alternative, various parametric approaches exist but generally introduce, among other hypotheses, the assumption of the normality of the risk factors and the existence of a linear relationship between these factors. Naturally, those assumptions are hardly sustainable. To take non-linearity into account, simulation approaches have been developed but could be time consuming and require a choice in the form of the distribution of the risk factors. All these approaches could be implemented taking the portfolio as a whole, and so assuming only one source of uncertainty, or the need to aggregate several risk factors.

The problem of integrating the dependency then arises. Copula functions, developed in the late 1950s by Sklar (1959), could be very useful in achieving this. Stated simply, copulas are functions that correspond to the dependence structure that exists between variables, from which we can obtain the joint distribution. First used in an insurance context, copulas have been of great interest in finance since the end of the 1990s (see Embrechts, McNeil, and Strauman 1999). Since that time, copulas have been applied to multiple areas of finance: pricing of derivatives (Cherubini and Luciano 2002, Cherubini and Luciano 2003b, Bennet and Kennedy 2004, etc.), a measure of contagion crisis (Costinot, Teiletche, and Roncalli 2000), credit risk (Li 2000, Schonbucher and Schubert 2001, Cherubini and Luciano 2003a, etc.) and portfolio allocation (Hennessy and Lapan 2002). Of course, the use of copulas for a market risk measure in general, and VaR in particular, has also been experimented (Cherubini and Luciano 2001, Rank 2002 or Ane and Kharoubi 2003). For a complete panorama of using copulas in finance one could refer to Jouanin, Riboulet, and Roncalli 2003 or Cherubini, Luciano, and Vecchiato 2004, among others. In the same vein, we develop a VaR analysis based on a copula approach in this paper. In this context, we aim to measure the accuracy of a portfolio’s VaR. However, unlike in previous work, we develop a trivariate risk factor analysis which could easily be extended to higher dimensions.

Our paper is organized as follows. In section 2 we introduce some notation and review basic facts about copulas, along with analysis of some better-known copulas. The second part is dedicated to the choice of copula that best fits our bivariate pairs and we extend those estimation procedures to the trivariate case. The third part consists of an empirical analysis of portfolio Value-at-Risk with the use of copulas. Part five concludes the paper.

2. Fundamentals of copulas
Copulas are functions that allow multivariate distribution functions to be obtained from the margins. In other words, we can say that every probability distribution function (pdf) could be obtained from a copula (characterizing the dependence structure) and the margins. In this section we will give the (simple) mathematical aspects that define the copula concept. Readers interested in the mathematics of copulas could refer to Joe (1997) or Nelsen (1999).
2.1 Definitions and properties

**Definition 1** A two-dimensional copula (or 2-copula) is a function $C : [0, 1]^2 \rightarrow [0, 1]$ with the following properties:

i) $C(u, 0) = 0 = C(0, u)$;

ii) $C(u, 1) = C(1, u) = u$ and $C(1, v) = C(v, 1) = v$;

iii) (2-increasing) For every $u_1, u_2, v_1, v_2$ in [0,1] such that $u_1 \leq u_2$ and $v_1 \leq v_2$, $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \leq 0$.

If the domain of $C$ is different from $l = [0, 1]$, but is defined on two non-empty subsets of $l$, containing 0 and 1, $C$ is called a sub-copula (2-dimensional in this case).

The use of copulas in finance is linked to their probabilistic interpretation (link between copulas and distribution functions of random variables) drawn from Sklar’s theorem.

**Theorem 1 (Sklar's theorem)** Let $F_1$ and $F_2$ be marginal cumulative distribution functions, and $Ran$ the range operator. Then for every $(x, y) \in \mathbb{R}^2$,

i) if $C$ is any sub-copula whose domain contains $RanF_1 \times RanF_2$, $C(F_1(x), F_2(y))$ is a joint distribution function with margins $F_1(x)$ and $F_2(y)$

ii) if $H(x, y)$ is a joint distribution function with margins $F_1(x)$ and $F_2(y)$, there exists a unique sub-copula $C$, with domain $RanF_1 \times RanF_2$, such that $H(x, y) = C(F_1(x), F_2(y))$.

If $F_1$ and $F_2$ are continuous, $C$ is a copula.

Sklar’s theorem allows every joint distribution function to be defined as a copula and two univariate cumulative distribution functions. Conversely we can express any copula with a joint distribution function and the “inverse” of two margins. That is the aim of the following corollary.

**Corollary 1** from the ii) of Sklar’s theorem, the unique sub-copula $C : RanF_1 \times RanF_2 \rightarrow I$ such that

$$H(x, y) = P(X \leq x, Y \leq y) = P(F_1(X) \leq F_1(x), F_1(Y) \leq F_1(y)) = C(F_1(x), F_2(y))$$

for every $(x, y) \in \mathbb{R}^2$ is

$$C(u, v) = P(F_1(X) \leq u, F_2(Y) \leq v) = P(X \leq F_1^{-1}(u), Y \leq F_2^{-1}(v)) = H(F_1^{-1}(u), F_2^{-1}(v)).$$

If $RanF_1 = RanF_2 = I$, $C$ is a copula.

This corollary is particularly useful as we are now able to construct new bivariate distributions using new marginals $G_1(x)$ and $G_2(y)$, which gives

$$G(x, y) = H(F_1^{-1}(G_1(x)), F_2^{-1}(G_2(y)).$$

As we can see in formula (2.1), copulas allow the construction of a bivariate (multivariate distribution to be split into two parts: the choice of the margins and of the copula function that merges them. All the preceding results lead to copulas being viewed as a dependence function. By analogy with the Pearson correlation coefficient, copulas can be used to define perfect positive or negative dependence as well as independence. This is the aim of the following theorem:
Theorem 2 (Fréchet-Hoeffding inequality) If \( C \) is a sub-copula, then for every \( (u, v) \) in \( \text{Dom} \ C \),

\[
\max (u + v - 1, 0) \leq C (u, v) \leq \min (u, v)
\]

(2.2)

This theorem is also true for copulas since every copula is also a sub-copula. The right-hand side (denoted by \( C^+ (u, v) \)) and the left-hand side (denoted by \( C^- (u, v) \)) of the inequality, represented in figures 1 and 2, are themselves copulas as they respect the three properties characterizing a copula function. All the dependence aspects are characterized by the choice of the specific copula function. The perfect dependence (positive or negative) is strongly linked to the concept of comonotonicity and countermonotonicity. Hoeffding and Fréchet have demonstrated that a perfect positive dependence (i.e. for any \( (x_1, x_2), (y_1, y_2) \), \( x_1 \leq y_1 \) and \( x_2 \leq y_2 \) or \( x_1 \geq y_1 \) and \( x_2 \geq y_2 \)) corresponds to the Fréchet upper bound \( C^- (u, v) \), and conversely for countermonotonicity. All copulas are included between those two bounds. For instance, the product copula (denoted \( C^+ (u, v) \)) represented in figure 3 corresponds to the situation where there is perfect independence between the two random variables considered:

\[
C^+ (u, v) = u.v
\]

By using Sklar’s theorem, we could restate (2.2) in terms of random variables and their distribution functions:

\[
\max (F_1 (x) + F_2 (y) - 1, 0) \leq H (x, y) \leq \min (F_1 (x), F_2 (y))
\]

Other properties of the copula could be useful in the risk management area. The first is that copulas respect the invariance property, i.e. the use of increasing transformations on each random variable leads to the same dependence structure (the same copula). Mathematically, we can write:

\[
H (\alpha x \leq x, \alpha y \leq y) = C (F_1 (x), F_2 (y)).
\]

This property will allow us to measure the aggregate risk of a portfolio.

Some financial applications involve determining the probability that the random variables are above a threshold. As before, we can use copulas to define joint survival functions. If we note \( \tilde{H} (x, y) \), the joint survival copula with two margins \( F_1 (x) \) and \( F_2 (y) \), we can write :

\[
\tilde{H} (x, y) = P (X > x, Y > y)
\]

\[
= P (X \leq x, Y \leq y) + P (Y > y) + P (X > x) - 1
\]

\[
= 1 - P (X \leq x) + 1 - P (Y \leq y) - 1 + P (X \leq x, Y \leq y)
\]

\[
= 1 - F_1 (x) - F_2 (y) + H (x, y).
\]

The last result could integrate the copula definition by use of (2.1):

\[
\tilde{H} (x, y) = 1 - F_1 (x) - F_2 (y) + C (F_1 (x), F_2 (y))
\]

\[
= F_1 (x) + F_2 (y) - 1 + C (1 - F_1 (x), 1 - F_2 (y))
\]

Figure 1: The minimum copula and its level curves.

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5 - It is usual to represent copulas with both a 3D plot of the function and also by use of level curves which correspond to the set of points of \( \mathcal{F} \) such that \( C (u, v) = c \) constant.

6 - This case corresponds to \( H (x, y) = f_1 (x) f_2 (y) \), i.e. the products of the cdfs. See below.
So if we define
\[ \tilde{C}(u, v) = u + v - 1 + C(1 - u, 1 - v), \] (2.3)
we have
\[ \tilde{H}(x, y) = \tilde{C}(F_1(x), F_2(y)), \]
where \( \tilde{C} \) is named the survival copula. This kind of copula could be useful for modelling some dependence structures, as we will see in the next part of this paper.

As for distributions, we could define the density of a copula which is given by:
\[ c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}. \]
Using this formula and Sklar's theorem, we could obtain the density of the joint distribution (canonical decomposition of the density) by multiplying the density of the copula under consideration by the densities of the marginals:
\[ f(u, v) = c(F_1(u), F_2(v)) f_1(u) f_2(v). \] (2.5)

We end this section by establishing a link between copulas and dependence in the tails of the distributions. Copulas could be considered as dependence measures. Nevertheless, as Venter (2001) has written, "Copulas differ not so much in the degree of association they provide, but rather in which part of the distributions the association is strongest". Copulas are therefore useful functions, particularly in financial risk management, or in an insurance context, where there is a need to "forecast the worst", to constitute a satisfying cushion to face the commitments. For example, imagine that a company is involved in two main activities. A crisis in both activities could lead to serious difficulties. Of course, one possibility is to use the diversification principle by increasing the number of activities. If we consider market risk, this involves an increase in the number of assets in a portfolio. Nevertheless, the use of the Markowitz framework is difficult to justify empirically (quadratic utility functions or lognormal distribution for the assets). Moreover, classic portfolio optimization uses the linear (Pearson) correlation coefficient as the measure of the dependence between the assets. However, this measure has serious shortcomings and cannot be recommended in all situations, i.e. in a non-elliptical world (see Embrechts, McNeil, and Strauman 2002). With this
goal in mind, alternatives have been developed. The main ones are the Kendall–τ and the Spearman–ρ. To measure the concordance specifically in the tails (both lower and upper), the tail dependence coefficient, \( \lambda \), has been developed. It corresponds to the probability of observing simultaneous extreme events (large or small) in all variables, and can be formulated with copulas, as a conditional probability:

\[
\lambda_u = \lim_{v \to 0^+} \frac{C(u, v)}{v},
\]

\[
\lambda_d = \lim_{v \to 1^-} \frac{C(1 - u, 1 - v)}{1 - v},
\]

in both cases, \( \lambda \) equal to zero corresponds to no tail dependence; in other words if \( \lambda = 0 \), extreme movements appear independently.

### 2.2 Common copulas

We have already analyzed three basic copulas (min, max and product). We now present several well-known copulas, most of which will be used in the empirical part of our study. We start with the simple construction of copula functions.

Unlike the traditional method for constructing joint distribution functions, we do not require the two margins to be of the same type. As we have seen, copulas are functions that bind all kinds of margins to form the multivariate distribution. So we only have to plug the desired margins into a copula function. Consider, for example, the bivariate Gaussian distribution. Its density is given by:

\[
\phi_r(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2 + \left( \frac{x_2 - \mu_2}{\sigma_2} \right)^2 - 2\rho \frac{(x_1 - \mu_1)(x_2 - \mu_2)}{\sigma_1\sigma_2} \right\},
\]

where \( x_1 \) and \( x_2 \) follow a normal distribution with mean \( \mu_1 \) and \( \mu_2 \) and standard deviation \( \alpha_1 \) and \( \alpha_2 \) respectively, and \( \rho \) is the correlation coefficient between the two variables.

From this density, we obtain the Gaussian copula:

\[
C^{Gauss}(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left\{ 2\rho st - s^2 - t^2 \right\} ds dt,
\]

where \( \Phi_0 \) stands for the bivariate standard normal cdf and \( \Phi \) the univariate standard normal cdf. Then we can construct the density of this copula by use of relation 2.4, which gives:

\[
\rho^{Gauss}(u, v) = \frac{1}{\sqrt{(1 - \rho^2)}} \exp\left\{ \frac{\Phi^{-1}(u)^2 + \Phi^{-1}(v)^2}{2} + 2\rho \Phi^{-1}(u) \Phi^{-1}(v) - \Phi^{-1}(u)^2 - \Phi^{-1}(v)^2 \right\}
\]

\[
= \frac{1}{\sqrt{|R|^{1/2}}} \exp\left\{-\frac{1}{2} \gamma^2 (\gamma^2 - 1)\right\},
\]

where \( I_\gamma = \gamma^2 - 1 \).
where $R$ is the correlation matrix, $I$ is the identity matrix and $\alpha = (\Phi^{-1}(u), \Phi^{-1}(v))'$.

The multivariate Gaussian distribution should not be used in a financial context even if it seems very appealing. Numerous empirical studies have shown that financial series exhibit skewed and fat-tailed distributions. However, as we have mentioned, the methodology allows margins different from Gaussians to be chosen to overcome this shortcoming.

As the copula reflects the dependence structure, we can use it to merge whatever distributions fit the empirical data. For example, figures 4 and 5 compare the density of the joint distribution function obtained from the use of the Gaussian copula with two Gaussian margins and with two $t$-student distributions with 3 degrees of freedom, and a correlation of 0.2.

It is important to emphasize the fact that the margins could be chosen not to be the same and so could be used to exhibit fat tails. Nevertheless, even if we use an extreme value distribution like the Fréchet or the Gumbel (see figure 6), the Gaussian copula does not display useful properties. This copula does not allow extreme co-movement to be taken into account, as it does not exhibit tail dependence unless the variables are perfectly correlated.

![Figure 5: Density and level curves of a distribution from a Gaussian copula and Student marginals with $\rho = 0.2$ and $\nu = 3$](image)

![Figure 6: Density and level curves of a distribution from an $t$-student copula and Gumbel marginals with $\rho = 0.2$](image)

To exhibit tail dependence, one possible alternative frequently used is to consider another elliptical copula, the Student copula, whose density is given by:

$$c_{\text{Student}} = |R|^{-1/2} \frac{\Gamma \left( \frac{\nu+1}{2} \right) \left[ \Gamma \left( \frac{\nu}{2} \right) \right]^N (1 + \frac{1}{\nu} R^{-1} \xi)^{-\frac{\nu+1}{2}}}{\left[ \Gamma \left( \frac{\nu+1}{2} \right) \left( \Gamma \left( \frac{\nu}{2} \right) \prod_{k=1}^N \left( 1 + \frac{s_k^2}{\nu} \right) \right)^{-1/2}}$$
where \( s_{x} = t_{x}^{-1}(u_{x}) \), \( \Gamma(. \) \) is the gamma distribution, and all the other variables are the same as those used before.

Even though the Student copula converges to the Gaussian one as the number of degrees of freedom increases, it generates fatter tails.

Nonetheless, this copula is symmetric, so the probability of obtaining an extreme loss is the same as that of obtaining an extreme gain, which contrasts with empirical observations, as financial assets’ return distributions are often left-skewed.

Theoretically, all the previous definitions could be generalized to higher dimensions. Unfortunately, it is not so easy in practice and most of the copulas are designed for bivariate distributions only. However, most of the financial applications of copulas require a lot of marginal distributions as there are numerous variables that have to be taken into account. We have written the multivariate form of the Gaussian and Student copulas (i.e. that could be used for a dimension higher than 2), but used them in the bivariate case for expository purposes (to obtain convenient graphs). Those copulas belong to the elliptical class of copulas but one other class of multivariate copulas is frequently used: Archimedean copulas. These copulas respect the following condition:

\[
C(u_{1}, ..., u_{n}) = \psi^{-1}(\psi(u_{1}) + ... + \psi(u_{2})) \quad \forall u_{i} \in [0, 1],
\]

with \( \psi \) a function called the generator of the copula, which satisfies:

\[\begin{align*}
\text{Definition 2:} & & i) & \psi(1) = 0 \\
& & ii) & \forall t \in [0, 1[, \ text{ } \psi'(t) < 0 \text{ (is increasing)} \\
& & iii) & \forall t \in [0, 1[, \ text{ } \psi''(t) \geq 0 \text{ (is convex)}. \\
\end{align*}\]

Three Archimedean copulas have been studied frequently:

- **Gumbel n-copula:** \( C(u_{1}, ..., u_{n}) = \exp \left\{ - \left[ \sum_{i=1}^{n} (-\ln u_{i})^{\alpha} \right]^{\frac{1}{\alpha}} \right\}, \text{ with } \alpha \geq 1. \)

- **Clayton n-copula:** \( C(u_{1}, ..., u_{n}) = \left[ \sum_{i=1}^{n} u_{i}^{-\alpha} - n + 1 \right]^{-\frac{1}{\alpha}}, \text{ with } \alpha \in [-1, 0] U [0, \infty[. \)

- **Frank n-copula:** \( C(u_{1}, ..., u_{n}) = -\frac{1}{\alpha} \ln \left\{ 1 + \left[ \frac{\prod_{i=1}^{n} (e^{-u_{i} \alpha} - 1)}{(e^{-u_{1} \alpha} - 1)^{n-1}} \right] \right\}, \text{ with } \alpha > 0. \)

In the three cases, the \( \alpha \) parameter controls the dependence. For example, the Gumbel copula can express perfect dependence when \( \alpha \rightarrow \infty \) and also independence with \( \alpha = 1 \), but not perfect negative dependence.

Even though several copulas that could meet our needs exist, the choice of the “right” copula is not so easy. A second difficulty arises with the choice of margins. As we have mentioned previously, the frequent use of the normal distribution is too far from reality to be useful in a risk management framework. Several distributions could be chosen to exhibit fat-tailed distributions, such as the Pareto distribution or the Student one. The choice of the copula function is clearly of prime importance. This is the aim of the following section.

### 3. Copula selection

In this study we use the daily log-returns of three indices from the French (CAC40), United States (S&P500) and Japanese (Nikkei) stock markets. The data we use comes from Fininfo and covers the period from 4 January 1991 to 25 January 2005 (3,633 days). We only keep account of data corresponding to common trading days in the three countries. This reduces our sample to 3,323 observations for each series. Figure 7 shows the evolution of the three indices relative to the prices on January 4, 1990. As we can see, the CAC40 and S&P500 seem to exhibit more comovements than
the Nikkei Index.

Table 1 contains the principal statistics of the series studied\textsuperscript{10}.

A simple look at table 1 calls to mind non-normal series. The skewness coefficients reveal asymmetry for all our series (negative for the CAC and the S\textsuperscript{t}iP500 and positive for the Nikkei), and the positive excess kurtosis coefficients for all our variables, with a maximum obtained for the S\textsuperscript{t}iP, confirms the non-normality of our data, as we can observe in figure 8, which gives a graphical representation of the return series as well as their empirical distribution (Epanechnikov kernel density) compared to a normal distribution. Of course the Jarque and Bera tests with values of $JB_{\text{rncac}} = 975.88$, $JB_{\text{risp500}} = 5181.70$ and $JB_{\text{rnnikkei}} = 846.40$ strongly reject the possible use of the Gaussian structure of our data and a normality assumption would generate inaccurate results.

In an initial attempt to consider the links between the different series we indicate the correlation matrix between the returns of the different indices.

\[
\begin{array}{cccc}
\text{CAC} & \text{S\textsuperscript{t}iP500} & \text{Nikkei} \\
\text{CAC} & 1 & 0.4420874 & 0.2471120 \\
\text{S\textsuperscript{t}iP500} & 0.4420874 & 1 & 0.1514724 \\
\text{Nikkei} & 0.2471120 & 0.1514724 & 1 \\
\end{array}
\]

We can see that the linear correlation is positive. The S\textsuperscript{t}iP500 and the Nikkei seem to be less dependent on each other than the CAC with those two indices. Nevertheless, as we mentioned previously, the correlation coefficient cannot offer a rigorous measure of the dependence between the series. In order to study the dependence structure of those indices more rigorously we will work in two steps. The first step will be dedicated to the bivariate relationships between the indices taken two by two. The second generalizes the previous results to the multivariate dependence context.

Several copulas could be used in order to characterize the dependency linking our variables. We have chosen to test one elliptical copula (Normal) and four Archimedean copulas (Gumbel, Frank, Clayton and its survival copula, the Heavy Right Tail copula) on our data. Two main reasons could justify the use of those copulas. The first is that they have been extensively studied and are particularly well known. The second is that those copulas have the particularity of being easily extensible to multivariate distributions, so they could be used to model our trivariate portfolio.

\textsuperscript{10} Results are given in percentage. $\mu_3$ corresponds to the skewness coefficient and $\mu_4$ to the kurtosis.
In the array below, we give the densities of the different Archimedean bivariate copulas \(^\text{11}\) obtained by use of equation 2.4 to the different copulas:

In the trivariate case, we only give the heavy right tail copula \(^\text{12}\) as it will be used to measure the structure of dependence between the three indices. For this copula, the generalization of (2.3) corresponds to:

\[
C(u, v, z) = -2 + u + v + z + C_{uv}(1 - u, 1 - v) + C_{uz}(1 - u, 1 - z) + C_{zw}(1 - v, 1 - z) - C(1 - u, 1 - v, 1 - z),
\]

with \(C_{uv}\) the two dimensional margin of the copula and \(C\) the trivariate Clayton copula. Thus, the density of this copula is given by:

\[
\frac{\partial^3 C(u, v, z)}{\partial u \partial v \partial z} = \left(\frac{-2a -1}{a}\right) \left(\frac{-a-1}{a}\right) a^2 (1 - u)^{-1 - a} (1 - v)^{-1 - a} (1 - z)^{-1 - a} \left(1 - z^{-1 - a} (1 - u)^{-a} + (1 - v)^{-a} + (1 - z)^{-a} - 2 \right)^{\frac{3}{2}}.
\]

Several approaches could be used to produce the appropriateness with empirical data. We have decided to focus on the Canonical Maximum Likelihood (CML) methodology of Romano (2002). Instead of estimating the multivariate distribution function 2.5 in a global approach, it can be

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\(^{11}\) - The density of the Gaussian copula was given in figure 6.

\(^{12}\) - The densities of all other trivariate copulas can be sent upon request.
less computationally intensive to separate the estimation of the copula and the margins. However, unlike the inference for margins (IFM) approach developed by Joe and Xu (1996), which assumes that we have specified parametric univariate marginal distributions, we do not have to make any assumption on the distributions as we use the empirical data and transform the sample data into uniform variates. This is what we did to transform the data used in figure 9 into the data in figure 10, which corresponds to the copulas of the three bivariate and the trivariate relationships. The different marginals have been reduced to uniform variables by using the empirical cumulative distribution function, i.e., the initial series \((x_i, y_i)\) are transformed into \((u_i, v_i)\) by taking \((u_i, v_i) = \left( \frac{\text{rank}(x_i)}{n+1}, \frac{\text{rank}(y_i)}{n+1} \right)\).

Other non-parametric methods, like the empirical method (Deheuvels 1978) or the kernel copula (Scaillet 2000) are presented with empirical applications in Cherubini, Luciano, and Vecchiato (2004).

As a second step we estimate the copula parameters via maximum likelihood estimation rather than least squares as it gives equal importance to all data, and not only where the maximum amount of data is, i.e., near the middle of the distribution.

\[
\log L(\theta) = \sum_{i=1}^{n} \log c(u_i, v_i) \\
\hat{\theta} = \arg \max_{\theta} \log L(\theta)
\]

with \(c(.)\) the density of the copula under consideration and \(\hat{\theta}\) the set of parameters to estimate.

![Figure 9: Empirical dependence of the three bivariate series and the multivariate one](image-url)
To choose the "right" copula we use several indicators. The first two are the value of the log likelihood function, to which we add the analysis of the tail concentration functions (left and right tails) and choose the copula which maximizes the different criteria. We do not explicitly display AIC or BIC goodness-of-fit tests as they will give the same result as the likelihood (only one parameter to estimate). We also add three other criteria based on a distance measure: the Anderson-Darling, the Integrated Anderson-Darling and a discrete $L^p$ norm, which measure the distance between the estimated copula and the empirical copula.\textsuperscript{14}

In fact, in the same vein of determining the univariate empirical cumulative distribution function, we can define the Deheuvels (1978) empirical copula which corresponds to a multivariate version of the empirical cumulative distribution function seen before:

\[
\hat{C}(t_1, \ldots, t_N) = \sum_{i=1}^N \prod_{j=1}^d \mathbb{1}_{r_{ij}^* \leq t_i},
\]  

where $r_{ij}^*$ corresponds to the rank statistics of the $j$th value of the $i$th variable, $t_i = 0, \ldots, n$ and $1$ the indicator function. We can also define the density of the empirical copula

\[
\hat{c}(t_1, \ldots, t_N) = \sum_{h=1}^2 \sum_{j=1}^d (-1)^{h+j} \hat{C}(t_1 \mathbb{1}_{\frac{h}{N} \leq t_1}, \ldots, t_N \mathbb{1}_{\frac{j-1}{N} \leq t_N} + 1).
\]

We can now compute the different distances between the parametric copula and the empirical one. The Anderson-Darling (AD) and Integrated Anderson-Darling (IAD) are given by the following equations:

\textsuperscript{14} We do not construct tests from those measures as it would be necessary to construct a test for each multivariate distribution given by the different copulas. For examples of how to construct those kinds of tests, see Malergerbe and Sornette (2003) for the Gaussian copula.
with $C_e(.)$ the parametric copula under estimation, all other notations remaining the same.

Both the AD and the IAD criteria are more concerned with deviations in the tail. Even if the AD criterion is well-known it has the drawback of being very sensitive to the presence of outliers (use of the max function). That is why we also use the IAD which is, in its discrete version, an average version of the AD test.

We also add a $L^p$ norm defined as:

$$L^2 = \sqrt{\sum_{N} \sum_{n=1}^{N} \left[ C_{e(n)} \left( \frac{t_1}{N}, \frac{t_2}{N} \right) - C_e \left( \frac{t_1}{N}, \frac{t_2}{N} \right) \right]^2}$$

These measures are frequently used in the literature (see Junker and May 2002 or Ane and Kharoubi 2003). Nevertheless, we must keep in mind that each of them could lead to different copulas being selected as they focus on various aspects of the goodness-of-fit. For example, the Anderson Darling and its cumulative function are concerned with the divergence in the tails, whereas the $L^p$ norm or the likelihood are more global. Lastly, like Olivier, Levi, and Carpenter (2003) in the insurance context, we use the tail concentration functions defined by Venter (2001). These functions return how much probability is in the upper or lower quadrant, i.e. in the region $< 0, 0 >, < 1, 1 >:

$$L(z) = P(U < z, V < z) / z^2 = C(z, z) / z^2$$

$$R(z) = P(U > z, V > z) / (1 - z)^2 = (1 + 2z + C(z, z)) / (1 - z)^2$$  \hfill (3.9)

$\forall z \in ]0, 1[$. Those two functions are of course closely linked to the concept of upper or lower tail dependence which are the cases corresponding to $z \to 1$ in $R(z)$ and $z \to 0$ in $L(z)$. They consist of a graphic tool that could be useful to visualize the convergence/divergence of the fitted copulas with the empirical one, by using the transformation explained on page 11.

Table 2 displays the parameters obtained for each group of returns via the CML method. The numbers in parentheses correspond to the standard errors of estimates. For each relationship under consideration, the highest log likelihood values are in bold.

<table>
<thead>
<tr>
<th>CAC/S&amp;P500/ Nikkei</th>
<th>Gaussian</th>
<th>Clayton</th>
<th>Frank</th>
<th>Gumbel</th>
<th>HRT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimates</td>
<td>0.4163019 (0.0127309)</td>
<td>0.5662758 (0.0152861)</td>
<td>2.5725728 (0.1109447)</td>
<td>1.359146 (0.1093865)</td>
<td>1.5784129 (0.0298984)</td>
</tr>
<tr>
<td>Likelihood</td>
<td>314.12</td>
<td>269.39</td>
<td>265.64</td>
<td>337.62</td>
<td>272.01</td>
</tr>
<tr>
<td>CAC/Nikkei</td>
<td>0.1570219 (0.0167811)</td>
<td>0.1783797 (0.0223268)</td>
<td>0.9415972 (0.0105976)</td>
<td>1.097254 (0.0128709)</td>
<td>1.0625632 (0.0225786)</td>
</tr>
<tr>
<td>Estimates</td>
<td>41.14</td>
<td>30.5</td>
<td>39.86</td>
<td>41.26</td>
<td>32.57</td>
</tr>
<tr>
<td>Likelihood</td>
<td>101.46</td>
<td>94.68</td>
<td>88.06</td>
<td>89.12</td>
<td>71.09</td>
</tr>
<tr>
<td>CAC/S&amp;P500/ Nikkei</td>
<td>0.3423602 (0.0150286)</td>
<td>0.2952071 (0.0146694)</td>
<td>1.4277123 (0.0107524)</td>
<td>1.158809 (0.0163186)</td>
<td>0.2694565 (0.0248522)</td>
</tr>
<tr>
<td>Estimates</td>
<td>101.46</td>
<td>94.68</td>
<td>88.06</td>
<td>89.12</td>
<td>71.09</td>
</tr>
<tr>
<td>Likelihood</td>
<td>422.13</td>
<td>299.60</td>
<td>271.78</td>
<td>276.38</td>
<td>273.3</td>
</tr>
</tbody>
</table>

Table 2: Estimated parameters with initial returns

15 - The errors are penalized more when they come from the extreme as the denominator of the AD and IAD measure is larger than in the middle of the distribution ($t_1 \to 0$ or $t_1 \to 1$).
As our series have a skewness different from zero, we also estimate the parameters of the copulas with use of negative returns (those returns are simply the opposites of the previous ones). Results are given in table 3. Of course, since the Gaussian and Frank copulas are symmetric in the bivariate case, they will display similar results if the distribution of the returns is also symmetric. As the skewnesses of the index return distributions are close to zero, we can verify that the estimated parameters in the case of negative log returns are similar to the results obtained in the previous table. We also use, in the bivariate case, a BB5 copula which corresponds in some way to an extension of the Gumbel copula but with two dependence parameters. We have not written the results as, for each pair of variables, we obtained the same results as those we obtained from the Gumbel copula (same likelihood, same α, highly non-significant δ). In figure 11 we have represented the different copula densities for the CAC/S&P500 data in the case of classic log returns. This figure confirms the symmetry of the Gaussian and Frank copulas.

Table 4 contains the values of the different goodness-of-fit indicators. Unlike Ane and Kharoubi (2003), who find that the Clayton copula is the best for capturing the dependence structure of bivariate stock index returns, we cannot obtain one single copula to best represent the empirical dependence. For the CAC40/S&P500, three copulas could claim to best represent the empirical data. The ML and Lp norm give the Gumbel copula as the best choice whereas the Anderson-Darling measure selects the Clayton copula and the IAD highlights the Gaussian copula. As the AD and IAD give more impact to the deviation in the extremes, whereas ML and Lp are more global measures, we can confirm that the Frank, the Gumbel and the HRT copula do not exhibit tail properties that suit our data well. This conclusion is less true for the negative returns, but still valid. For the CAC40/Nikkei dependence, the Gaussian copula appears to constitute the best fit all along the empirical distribution, whereas if we pay more attention to the tails, it is the Clayton copula that best fits the data. For negative returns, the Gumbel copula offers an alternative to the Gaussian if we consider the complete distribution. The last pair under consideration is the S&P500/Nikkei. In this case, each distance measure selects a different copula for positive returns whereas the Gaussian copula seems to be the best for 3 of the 4 criteria for negative returns. These results are in line with those obtained by

Figure 11: Estimated parametric copulas for the CAC40/S&P500 pair

---

16 - This copula could lead to the Gumbel copula (δ → 0) as well as the Galambos copula (α = 1).
17 - We have not plotted the HRT copula density as it corresponds to the symmetric of the Clayton copula.
Malevergne and Sornette (2003) who found that the Gaussian copula should not be rejected in 70% of the cases they studied. The copula which performs the worst is the Frank copula.

We only use a likelihood estimate in the trivariate case as the distance measures cannot be reasonably computed. In fact, we would have to draw 33,233 iterations to obtain the AD, IAD or D measures. The concentration tails for the CAC/S&P500 pair have been plotted in figure 12. We can verify that the two concentration functions tend towards 1. As we wrote before, the Gumbel copula is the best one in the right tail (near (1,1)). However, it is the Frank copula which appears to best fit the empirical data in the right tail. This result could be surprising as the previous results show that it always underperforms the other copulas. Nonetheless, we can easily explain this result by the fact that the previous fitting distance measures were global. Figure 13 represents the different level curves implied by each copula under consideration for the CAC/S&P500 pair.

We should note that we have obtained the different parameters by a maximum likelihood estimation. In order to choose the best copula in a specific area (for example in the tails), it is also possible to minimize the distance in this part of the distribution. For example one can obtain the parameter estimates by minimizing the IAD distance, or the distance between the R or L measures defined in equation 3.9 with empirical data. This methodology brings more accuracy into the specific area under consideration, but could lead to a loss of information. This could have, for example, a significant impact on the calculation of the Value-at-Risk.

We can now test the contribution of copulas to the market risk assessment. We have chosen to select the VaR to perform our empirical analysis of copula use in a risk management context. Of course, the Valueat-Risk is a far-from-perfect measure, but it has constituted a reference since 1993. Several methods have been developed for obtaining the VaR. Even if there is no one panacea we have chosen to use a Monte-Carlo approach because it has fewer assumptions than the other classic approaches. Moreover, it allows one to benefit from the use of copulas. This constitutes the core of the following section.

### Table 3: Estimated parameters with opposite returns

<table>
<thead>
<tr>
<th></th>
<th>Gaussian</th>
<th>Clayton</th>
<th>Frank</th>
<th>Gumbel</th>
<th>HRT</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Estimates</strong></td>
<td>0.4163019</td>
<td>0.57848565</td>
<td>2.5725744</td>
<td>1.35412005</td>
<td>0.5852758</td>
</tr>
<tr>
<td></td>
<td>(0.01327359)</td>
<td>(0.05268837)</td>
<td>(0.011566447)</td>
<td>(0.01177115)</td>
<td>(0.02986518)</td>
</tr>
<tr>
<td><strong>Likelihood</strong></td>
<td>314.12</td>
<td>272.01</td>
<td>205.65</td>
<td>328.93</td>
<td>209.39</td>
</tr>
<tr>
<td><strong>Estimates</strong></td>
<td>0.15702194</td>
<td>0.16256333</td>
<td>0.9145972</td>
<td>1.09007675</td>
<td>0.1730797</td>
</tr>
<tr>
<td></td>
<td>(0.01678121)</td>
<td>(0.02227705)</td>
<td>(0.01596076)</td>
<td>(0.01292124)</td>
<td>(0.0227268)</td>
</tr>
<tr>
<td><strong>Likelihood</strong></td>
<td>41.14</td>
<td>32.47</td>
<td>30.49</td>
<td>40.90</td>
<td>36.50</td>
</tr>
<tr>
<td><strong>Estimates</strong></td>
<td>0.24434197</td>
<td>0.25536537</td>
<td>1.42777121</td>
<td>1.17248554</td>
<td>0.2953071</td>
</tr>
<tr>
<td></td>
<td>(0.01585001)</td>
<td>(0.02418545)</td>
<td>(0.01073235)</td>
<td>(0.01440025)</td>
<td>(0.02669104)</td>
</tr>
<tr>
<td><strong>Likelihood</strong></td>
<td>101.46</td>
<td>71.09</td>
<td>88.06</td>
<td>107.19</td>
<td>94.68</td>
</tr>
<tr>
<td><strong>Estimates</strong></td>
<td>0.11335</td>
<td>0.3129753</td>
<td>1.484478</td>
<td>1.178346</td>
<td>0.32776</td>
</tr>
<tr>
<td></td>
<td>(0.02271792)</td>
<td>(0.01629872)</td>
<td>(0.01092148)</td>
<td>(0.00882127)</td>
<td>(0.01611308)</td>
</tr>
<tr>
<td><strong>Likelihood</strong></td>
<td>422.13</td>
<td>273.30</td>
<td>269.95</td>
<td>285.92</td>
<td>299.6</td>
</tr>
</tbody>
</table>

4. Copulas and Value-at-Risk

For simplicity, the VaR is quite often calculated, in the literature, in a univariate framework. Although this can be justified for passive management, it could lead to short-sightedness. In a multivariate framework we need to measure the dependence between the different risk factors or assets that make up the portfolio. In this context, one of the first attempts was proposed by Longin (2000) who develops an extreme value approach for VaR estimation in univariate and multivariate contexts. Nevertheless, the VaR aggregation proposed in this paper is built on a classic normality hypothesis. However, the traditional use of the correlation coefficient should be used only in the case of elliptic
distributions. In all other cases, it could lead to incorrect results. To assess the advantages given by copulas in obtaining the VaR, we propose to backtest the validity of portfolio VaR calculated from a multivariate distribution given by our copula functions.

### 4.1 Multivariate simulation

Multivariate dependence could be simulated in several ways. Here, we will focus only on the most general approach. Simulating dependent variables involves simulating two or more variables whose joint distribution is a copula. Let’s begin with the bivariate case. We have to draw two \([0,1]\) uniformly distributed independent random variables \(U\) and \(W\). Then we use the conditional sampling to create a new series with the dependence structure expressed by the copula function. This conditional distribution from which we start is given by:

\[
V = c_{uv}(U, W)
\]

(4.10)

The second step is to compute the inverse function of equation 4.10. By taking \(c_{uv}^{-1}(w)\), we obtain a new series \((v)\) which exhibits the required dependence. To obtain the desired pairs with the copula dependence structure, we then have to take the inverse of the cumulative density function of the two variables. Of course, this supposes that we know the form of the density function for each margin. Consider, for example, the Clayton copula. The conditional distribution corresponding to the bivariate case of the Clayton-\(n\) density is:

\[
\frac{\partial C(u, v)}{\partial u} = u^{-a-1}(u^{-a} + v^{-a} - 1)^{-\frac{1}{a} - 1}.
\]

<table>
<thead>
<tr>
<th></th>
<th>Positive returns</th>
<th>Negative returns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ML</td>
<td>AD</td>
</tr>
<tr>
<td>CAC/SP500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gaussian</td>
<td>314.12</td>
<td>0.0536964</td>
</tr>
<tr>
<td>Clayton</td>
<td>360.39</td>
<td>0.0637311</td>
</tr>
<tr>
<td>Frank</td>
<td>265.62</td>
<td>0.2045733</td>
</tr>
<tr>
<td>Gumbel</td>
<td>337.82</td>
<td>0.1504751</td>
</tr>
<tr>
<td>HRT</td>
<td>272.61</td>
<td>0.5891048</td>
</tr>
<tr>
<td>CAC/Nikkei</td>
<td>101.46</td>
<td>0.0923757</td>
</tr>
<tr>
<td>Gaussian</td>
<td>91.98</td>
<td>0.0412147</td>
</tr>
<tr>
<td>Clayton</td>
<td>88.00</td>
<td>0.1825152</td>
</tr>
<tr>
<td>Frank</td>
<td>89.12</td>
<td>0.1947349</td>
</tr>
<tr>
<td>HRT</td>
<td>71.00</td>
<td>0.1424787</td>
</tr>
</tbody>
</table>

**Table 4:** Results of the distance measures for the positive and negative returns

![Left concentration tail](image1.png) ![Right concentration tail](image2.png)

**Figure 12:** Left and right concentration tails for the CAC/SP500 pair
We only have to solve \( \frac{\partial C(u,v)}{\partial u} = w \) for \( v \) to obtain

\[
v = \left( u^{-\alpha} \left( w^{-\frac{1}{1+\alpha}} - 1 \right) + 1 \right)^{-\frac{1}{\alpha}}.
\]

Of course we only need to generalize this approach in the trivariate case. For the Frank-3 copula, we first have to simulate two uniform random variables, as in the bivariate case. From these, we compute the inverse of the bivariate conditional distribution to create a new variable with a dependency given by the Frank-2 copula.

\[
v = -\frac{1}{\alpha} \log \left[ 1 - \frac{(1 - e^{-\alpha})}{1 + (w^{-1} - 1)e^{-\alpha u}} \right].
\]

We reiterate the methodology to generate a variable \( z \), conditional on the previous:

\[
C_3(z|u, v) = \frac{\partial^2 C_3(u, v, z)}{\partial u \partial v} / \left( \frac{\partial^2 C_2(u, v)}{\partial u \partial v} \right),
\]

and then inverse this function to obtain \( z \).

---

**Figure 13**: Spread between the different estimated parametric copulas and the empirical one for the CAC30/ST&IP500 pair.
When it is not possible to inverse the conditional distribution, as in the Gaussian or Gumbel cases for example, we need to use a numerical procedure.

We can generalize this methodology to the trivariate case. For example, if we consider the trivariate Clayton copula, we need to proceed in two steps. As before, we have to simulate three uniform random variables (let’s use $w_1$, $w_2$, $w_3$). Setting $u_1 = w_1$, we then have to simulate a first random variable, $u_2$, conditional on $u_1$:

$$w_2 = \frac{\frac{\partial C_2(u_1, u_2)}{\partial u_1}}{\frac{\partial C_1(u_1)}{\partial u_1}} = \left[u_1 \cdot (-1 + u_1^{-\alpha} + u_2^{-\alpha})\right]^{-1-\frac{1}{\alpha}}$$

and solve for $u_2$. We obtain:

$$u_2 = \left[\frac{-\frac{1}{1+\alpha} + 1 - u_1^{-\alpha}}{u_1^{\alpha}}\right]^{-1/\alpha}$$

As a second step we need to obtain a third random variable $u_3$ conditional on $u_1$ and $u_2$:

$$w_3 = \frac{\frac{\partial^2 C_3(u_1, u_2, u_3)}{\partial u_1 \partial u_2}}{\frac{\partial^2 C_1(u_1, u_2)}{\partial u_1 \partial u_2}} = \left[\frac{-1+u_1^{-\alpha}+u_2^{-\alpha}}{-2+u_1^{-\alpha}+u_2^{-\alpha}+u_3^{-\alpha}}\right]^{2+\frac{1}{\alpha}}$$

and solve for $u_3$ which results in

$$u_3 = \left(\frac{-1+u_1^{-\alpha}+u_2^{-\alpha}}{w_3^{2\alpha+1}} + 2 - u_1^{-\alpha} - u_2^{-\alpha}\right)^{-\frac{1}{\alpha}}$$

In figure 14, we represented the influence of an increase in the dependence parameter for the dual of the Clayton copula (HRT) in the trivariate case.
4.2 VaR analysis

We now turn to a practical use of copulas in risk management. To verify which copulas can provide the best VaR estimations, we use a backtesting methodology of the VaR computed by Monte Carlo simulations.\(^\text{18}\) We generate a large number of pairs (200,000 runs) of risk factors governing our portfolios. To simplify, we suppose that they are composed of 2 or 3 equally-invested indexes. We develop two possibilities. The first is to estimate the parameters from the whole sample period. This allows the maximum amount of observed data to be taken and therefore the maximum number of possible extreme values to be integrated. Unfortunately, this is less useful when the parameters are not stable. The second approach consists of using a one-year (250-day) window of data to obtain the margin and copula parameters. From these we generate a large number of scenarios to compute the VaR. Then we roll over the procedure but with a one-day time lag.

After generating the simulations we can use three possibilities to compute the VaR. The first is to take the 1st quantile of the possible returns of the simulated portfolio.\(^\text{19}\) That is what we do. We could also use a parametric distribution to fit the simulated returns to obtain the VaR or to estimate the tail of the distribution with an extreme value distribution. Different marginal distributions are used to move from the uniform variables to the simulated returns of the indices. After a maximum likelihood analysis we have selected only four distributions: the Student-t and Skewed Student-t (best fit of the empirical returns for the three indices), the Gaussian as it is frequently assumed in numerous financial studies, and the empirical one to allow for comparison between the different parametric marginals.

If we consider the traditional returns\(^\text{20}\) for which the results are indicated in the upper part of table 5, the Clayton copula gives the best results. As we have used 3,323 returns for each series, we expect fewer than 34 cases where the returns of our portfolio are below the VaR. We can see that the Clayton copula respects this condition the most. For the CAC/Nikkei, both the Gaussian and Clayton copulas respect the hypothesis and for the S&P500/Nikkei the Frank copula offers the same conclusion. Finally, the Gumbel copula and the HRT copula never allow the boundary condition corresponding to the risk level we have chosen to be respected. The second lesson is that Gaussian marginals always give the worst results, whereas the Skew-t distribution is very close to the results obtained with the empirical distributions and offers greater flexibility than the classic Student-t distribution. As expected, we obtain better results with negative returns, except for the Clayton copula. As before, the HRT copula, as it is the dual of the Clayton copula, offers the same conclusion as for the traditional returns. We can note that the Gumbel copula respects the limit in two cases out of three and is very close for the CAC/S&P500. For the portfolio based on CAC/Nikkei we respect the limit in 3 cases out of 5, whereas all but one copulas lead to the expected results in the S&P500/Nikkei case. For the rest of our paper we will use negative returns as they seem to give better global results.\(^\text{21}\) Before continuing, we can remark that not one of the goodness-of-fit tests we used clearly reveals the role that the HRT/Clayton copula could play in our risk management study. A second problem needs to be solved. From our methodology we have estimated the various VaR measures from all the data. This can lead to a major drawback as the distribution could change. We can now check our previous results with 250-day rolling windows.\(^\text{22}\) Figure 15 shows the evolution of the different copula parameters if we estimate them on a rolling window. As we can see, the high range of the parameters does not allow us to assume the stability of the process. It could also be of interest to note the link between the movement of the parameters and the evolution of the indices in figure 7. We can observe that the dependency has increased in bear markets but tends to decrease in bull markets. We have again estimated the different VaR numbers in the case of a HRT copula and a Gaussian copula for the negative log returns, with 250-day rolling windows. Figure 16 represents the backtesting for VaR with non-central Student margins. In this case, we obtain one excess less than before but we still have 3 excesses over the admissible limit of 31.\(^\text{23}\) This figure shows the superiority of the HRT copula over the Gaussian hypothesis. We also see that the amount of capital required

---

\(^{18}\) Of course, our methodology is still valid for any other risk measure.

\(^{19}\) Or the 99th quantile if we use opposite returns (then a gain corresponds to a negative loss).

\(^{20}\) By traditional returns or classic returns, we mean \(\log(P_t/P_{t-1})\), whereas negative log returns correspond to the opposite of the previous formula.

\(^{21}\) It would be the same if we also add a test of the dual of the Gumbel copula.

\(^{22}\) This is the minimum number of consecutive observations that the banks need to use in their VaR estimation.

\(^{23}\) As we have only 3,023 out-of-sample data.
with the Value-at-Risk calculated from this dependency structure is less than before and so is more in adequation with banks' needs. For all the other cases we obtain similar conclusions but with VaR which respect the threshold of no more than 31 excesses.

Figure 15: Evolution of the different copula parameters for the CAC/S&uuml;P500 pair

Figure 16: Backtesting of VaR for CAC/S&uuml;P500 with rolling windows of 250 days
The next step in our study concerns the analysis of the VaR for the HRT copula in the trivariate case. We obtain only 31 threshold excesses with the trivariate HRT copula and skew-t margins, and 43 for the Gaussian copula, which again confirms that the HRT copula fits the empirical dependency of our data well. Lastly, we have re-estimated the VaR number with floating parameters. We only obtain 28 excesses over the threshold, as can be seen in figure 17.

![Figure 17: Backtesting VaR measures in the trivariate case](image)

All the previous results we obtained allow us to select the HRT copula as the best choice in a risk management context.

<table>
<thead>
<tr>
<th>Method</th>
<th>Student-t</th>
<th>Skew-t</th>
<th>Normal</th>
<th>Empirical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>46</td>
<td>43</td>
<td>98</td>
<td>44</td>
</tr>
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<td>Gaussian</td>
<td>41</td>
<td>38</td>
<td>54</td>
<td>38</td>
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<tr>
<td>Frank</td>
<td>46</td>
<td>43</td>
<td>98</td>
<td>45</td>
</tr>
<tr>
<td>Clayton</td>
<td>41</td>
<td>38</td>
<td>54</td>
<td>39</td>
</tr>
<tr>
<td>HRT</td>
<td>46</td>
<td>43</td>
<td>98</td>
<td>46</td>
</tr>
</tbody>
</table>

Table 5: Backtesting VaR in the trivariate case, under various marginals and copula functions - classic returns (top table) and negative returns (bottom table)
5. Conclusion

The uses of Value-at-Risk are numerous and always have a strategic impact. This is the case for regulators and investment banks for whom the VaR determination is of particular interest. Nevertheless, most of the time VaR is calculated with strong assumptions (which are more or less explicit) contrasting with empirical observations. Even if an alternative to the assumption of normality is used, the analysis is often carried out in a univariate framework. However, it is hard to assume that the books of the different trading desks are static. Even in the case of portfolio management, the different portfolios are subject to rebalancing, minimizing the advantage gained from the simplification of reality. In this paper we develop a multivariate framework for risk measure computation. We have developed an empirical analysis of the use of several copulas in obtaining the Value-at-Risk of a portfolio with several risk factors. We have computed the VaR from five copulas and four margins (analytic as well as non-parametric). Our bivariate tests realized from a Monte-Carlo analysis show that the Heavy-Right Tail copula, which is the dual copula of the Clayton one, always respects the validity condition of our backtests. The extension of this bivariate copula to the multivariate (trivariate case) we propose generates the same conclusion.

The methodology developed in this paper could of course be adapted to the pricing of complex financial products with multiple non-independent underlying risk factors.

For further developments, it would be interesting to extend our work to a more complex portfolio structure (such as, for example, long and short positions, various asset classes, etc.), and to various confidence levels (95% and 99.9%). We could also verify whether there is a best height of the window used for parameter estimation. The methodology we proposed also allows us to extend our work to other risk measures, such as Expected Shortfall, and could easily be adapted to higher dimensions. Finally, we could measure the impact of copula VaR optimization on portfolio selection.
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