Dynamic Asset Pricing Theory with Uncertain Time-Horizon

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Abstract
This paper addresses the problem of pricing and hedging a random cash-flow received at a random date in a general stochastic environment. We first argue that specific timing risk is induced by the presence of an uncertain time-horizon if and only if the random time under consideration is not a stopping time of the filtration generated by prices of traded assets. In that context, we provide an explicit characterisation of the set of equivalent martingale measures, as well as a necessary and sufficient condition for a convenient separation between adjustment for market risk and timing risk. We also present price bounds consistent with perfect replication in the absence of arbitrage for an asset paying off a random amount at a random time. As is often the case, such bounds are actually too wide to be of any practical use and we consider several choices (minimal martingale measure, minimum entropy measure) for narrowing down to one the number of equivalent martingale measures.

Key words: asset pricing, uncertain time-horizon, random time, incomplete markets.
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EDHEC pursues an active research policy in the field of finance. EDHEC-Risk Institute carries out numerous research programmes in the areas of asset allocation and risk management in both the traditional and alternative investment universes.
The theory of asset pricing in multi-period settings under uncertainty is now relatively well understood. One of the most spectacular achievements of that theory is to provide, under suitable assumptions, a unified framework for the valuation of uncertain and delayed cash-flows, with direct implications for the optimal behaviour of the firms and the investors. In particular, it is known that the absence of arbitrage is essentially equivalent to the existence of an equivalent martingale measure (EMM), under which discounted prices are martingales (Ross (1978), Harrison and Kreps (1979), Harrison and Pliska (1981)). A particularly successful application of that theory to the relative pricing of a redundant asset is the celebrated Black-Scholes (1973) option pricing formula. Arbitrage pricing theory is completed by equilibrium models which provide useful insights into an understanding of primitive security prices by specifying a pricing kernel expressed in terms of agents' preferences. Following the single-period Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965), equilibrium asset pricing models have been developed in multi-period settings in both discrete time (Rubinstein (1976), Breeden and Litzenberger (1978)) and continuous time (Merton (1973), Breeden (1979)). Hence, it seems that financial economists are well-equipped, at least in theory, for handling the valuation of any type of cash-flow.

Given that an investment horizon is in practice not frequently known with certainty at the date of investment, it seems in particular desirable to relax the restrictive assumption that the timing of cash-flows on assets is known with certainty ex-ante. Natural examples abound in the context of real option pricing (see Dixit and Pindyck (1993) for a survey). An assumption often made in that literature is that a company's investment opportunity has a known, or even infinite, time to maturity. While such an assumption allows the computation of option value to be greatly simplified, it is a clear distortion of reality. For example, the time to maturity could be the time before a competitor enters the market with a similar or substitute project, or the time before a technological change makes the company's product obsolete. In general, this date is not known ex-ante and depends upon the value of the underlying investment opportunity, since competition is enhanced as the investment opportunity gets more profitable. Other examples include pricing and hedging of securities with embedded prepayment options (e.g. mortgage-backed securities, convertible and callable bonds), valuation of catastrophe insurance contracts, among others. Another typical application is optimal investment and consumption when an agent is faced with an uncertain date of death or retirement. Also related to portfolio problems is the example of an investor who believes that he or she has better access to information about a stock, or a better ability to process it, than the market, i.e. the investor could detect non-zero alphas, but is uncertain about when this information will be impounded in market prices.

Various attempts at dealing with the problem of pricing a cash-flow received at a random date in a situation where this random date introduces some new uncertainty into the economy can be found in the literature. An early example is a paper by Brennan and Schwartz (1976), who provide a pricing formula for an "equity-based" life insurance contract by computing expectations with respect to an investor's uncertain date of death under the original measure. Formally related is a paper by Carr (1998), where the author introduces a convenient method for pricing American options by fictitiously allowing the maturity date of the contract to be random, and then letting the variance of the random time go to zero. More directly related is some recent research on the so-called reduced-form approach to credit risk, where potential default induces some uncertainty about the timing of the cash-flow to a corporate bondholder (see for example Duffie and Lando (2000), Duffie and Singleton (1999), Duffie, Schroder and Skiadas (1996), Elliott, Jeanblanc, and Yor (2000), Jarrow and Turnbull (1995), Lando (1997), Madan and Unal (1998), among many others). While there seems to exist an abundant research on various applications of the problem, no systematic and general framework has been developed, however, for asset pricing in the presence of a random date of maturity.

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1 - No arbitrage and existence of an EMM are equivalent in the finite-dimensional case. Further technical conditions need to be imposed in the infinite-dimensional case to ensure that the absence of arbitrage implies the existence of an EMM (see for example Duffie (2001)).

2 - When the uncertain time τ is a stopping time of the filtration generated by past values of asset prices, no new uncertainty is added to the economy and standard tools may be used to solve dynamic valuation problems (see section 1). Examples are the pricing of American or barrier options.

3 - In a "structured" approach to credit risk, time of default is a stopping time of the filtration generated by the value of the assets of the firm (see Merton (1974), Black and Cox (1976) or Longstaff and Schwartz (1995) for example). The theory developed in this paper is not needed in this case (see section 1).
of timing uncertainty. This paper is an attempt to fill in that gap. It builds upon the aforementioned research, which it extends in several important directions. In a nutshell, we provide in this paper a set of conditions under which timing risk exists, and can be priced. Our results are as follows. First of all, specific timing risk is induced by the presence of an uncertain time-horizon whenever the random time under consideration is not a stopping time of the filtration generated by prices of traded assets (see Section 1 for an economic interpretation). In this context, uncertainty over an investor’s time-horizon induces a specific form of market incompleteness, and we provide an explicit characterisation of the set of equivalent martingale measures (equation (3)) in such an economy. We also provide a necessary and sufficient condition (assumption 3) for a convenient separation between adjustments for market risk and timing risk, and discuss several ways of narrowing down the number of admissible equivalent martingale measures. These results allow us to obtain general pricing formulae and explicit hedging strategies for random cash-flows in the presence of timing risk.

The paper is organised as follows. In section 1, we introduce a general model of an uncertain time-horizon. Sections 2 and 3 are devoted, respectively, to dynamic arbitrage pricing and hedging of assets paying off random cash-flows at random dates. A conclusion can be found in Section 4, while proofs of some results and technical details are relegated to the Appendix.

1. The Economy

In this section, we introduce a general model for the economy in the presence of an uncertain time-horizon. The underlying uncertainty is modeled by the complete probability space (, , ) on which is defined a one-dimensional standard Brownian motion W. The state space (, , ) is endowed with a filtration satisfying the usual conditions which is the usual augmentation of the filtration generated the standard Brownian motion W. All statements involving random variables are understood to hold either almost everywhere or almost surely depending on the context. We denote by the expectation under the probability measure of a random variable X conditional upon information available at time t.

1.1 Financial Market Model

Of particular interest is some risky asset, the price of which, denoted by , is assumed to be given by

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t, \quad S_0 = 1
\]  

(1)

The extension to the case of many assets would be straightforward. We further assume that a risk-free asset is also traded in the economy. The return on that asset, typically a default free bond, is given by \( \frac{dB_t}{B_t} = r_t dt \) and \( B_0 = 1 \), where \( r_t \) is the risk-free rate in the economy. Throughout this paper, we assume that:

Assumption 1. The coefficients \( u_t, r_t \) are bounded and deterministic functions of time and \( r_t \geq 0 \).

Assumption 2. The coefficient \( \sigma_t \) is a bounded, invertible, deterministic function of time and the inverse \( \sigma_t^{-1} \) is also a bounded function.

Under these assumptions, the market is complete and arbitrage-free (see for example Karatzas (1996)). Starting from a complete market situation allows us to more easily focus on the specific form of incompleteness induced by uncertainty over time-horizon. We denote by \( Q_0 \) the equivalent martingale measure with a Radon-Nikodym density with respect to \( \mathbb{P} \), \( \xi_0 \), given as the solution to \( d\xi_0(t) = \xi_0(t) \beta_t dW_t \), where \( \beta_t = \sigma_t^{-1}(u_t - r_t) \).

4. On the other hand, the question of portfolio selection under an uncertain time-horizon has already received significant attention, and we refer the reader to Markowitz and Uroševic (2005) for the static case, Yan (1966) and Hakansson (1969, 1971) for the dynamic case in discrete-time, as well as Merton (1971), Richard (1975) and Blanchet-Scaillet et al. (2001) for the dynamic case in continuous-time (see also Lynch (2000), Louri and Poncet (2001) or Cvitanic et al. (2002) for examples among many others of optimal asset allocation models that exhibit an explicit dependency in the time-horizon in the presence of stochastic opportunity sets driven by predictability in return, stochastic interest rates, or dynamic learning, respectively).

1.2 Random Time Model
We focus on the following valuation problem: pricing a contingent claim with payoff \( X_\tau = \Phi(S_\tau) \) at some random date \( \tau \), for some regular function \( \Phi(\cdot) \). A first example is \( \Phi(x) = \max(x - K, 0) \), the pay-off of an European call on \( S \) with random maturity \( \tau \) and strike price \( K \), e.g., an employee stock option or a real option.\(^6\) Another example is \( \Phi(x) = x \), the payoff of an insurance contract on the value of the underlying asset (e.g. a catastrophe insurance contract on the value of an insured property). A last example is \( \Phi(x) = K \), where \( K \) is a real positive number, the pay-off of some non-contingent insurance contract on \( \tau \), e.g. a life insurance contract paying off a fixed amount on the date of the insuree’s death.

It is important to note that we do not assume that \( \tau \) is a stopping time of the filtration \( \mathcal{F} \) generated by asset prices. The date \( \tau \) is only taken to be a positive random variable measurable with respect to the sigma-algebra \( \mathcal{A} \). In other words, we do not assume that observing asset prices up to date \( \tau \) implies full knowledge about whether \( \tau \) has occurred or not by time \( t \). Formally, it means that there are some dates \( t \geq 0 \) such that the event \( (t < \tau) \) is not \( \mathcal{F}_t \)-measurable. When \( \tau \) is a \( \mathcal{F} \)-stopping time, e.g. the first hitting time of a deterministic barrier by asset prices, it is possible, although sometimes difficult, to apply the standard tools of dynamic valuation and optimisation problems. Examples are the pricing of American or barrier options. In this paper, we are instead interested in situations such that the presence of an uncertain time-horizon induces some new uncertainty in the economy.

There are two sources of uncertainty related to optimal investment in the presence of an uncertain time-horizon, one stemming from the randomness of prices (market risk), the other stemming from the randomness of the timing of exit \( \tau \) (timing risk). A serious complication is that, in general, these two sources of uncertainty are not independent. Separating out these two sources of uncertainty is a useful operation that may be achieved as follows. Conditioning upon \( \tau \) allows one to isolate a pure asset price uncertainty component: given a specific realisation of \( \tau \), the only remaining source of randomness comes from asset prices. On the other hand, conditioning upon \( \mathcal{F}_\infty \) allows one to isolate a pure timing uncertainty component. Since \( \mathcal{F}_\infty \) contains information about the whole path of risky asset prices, \( \mathbb{P}[\tau > t | \mathcal{F}_\infty] \), for example, is the conditional probability that the event of interest is still to happen at date \( t \) given all possible information about asset prices. Some assumption is needed at this point to specify the exact nature of the relationship between asset price uncertainty and timing uncertainty. An extreme assumption consists in taking \( \mathbb{P}[\tau > t | \mathcal{F}_\infty] = \mathbb{P}[\tau > t] \) for all \( t \). This is an independence assumption, which expresses that the timing of the event of interest is totally unrelated to asset prices. Such an assumption is a clear oversimplification since there exist a variety of situations (real option, employee stock option, mortgage backed-security, etc.) where some dependence of the event of interest upon risky asset prices naturally comes into play. In this paper, we instead make the following more general assumption on the conditional distribution of \( \tau \) given \( \mathcal{F}_\infty \). That assumption is known as the \( K \)-assumption in probability theory (see for example Mazziotto and Szpirglas (1979)).

**Assumption 3.**

\[
\mathbb{P}[\tau > t | \mathcal{F}_\infty] = \mathbb{P}[\tau > t | \mathcal{F}_t] \tag{2}
\]

Despite its technical character, assumption 3 is a very natural assumption, the interpretation of which is as follows. It requires that the probability of the event happening or not before time \( t \) does not depend upon knowledge about the whole asset return path (captured by \( \mathcal{F}_\infty \)), including what happens after time \( t \), but rather solely upon knowledge about asset returns up to time \( t \) (captured by \( \mathcal{F}_t \)). The important feature is that past asset prices, and not future asset prices, may affect uncertainty about the timing of the event of interest. Under this formulation, the \( K \)-assumption appears as a desired feature in most reasonable financial context, ruling out at most features such as inside information.

\(^6\) In general, employee stock options and real options are American options, and considering them as European options is only an approximation, except if the underlying asset pays no dividends, in which case early exercise is not optimal.
For tractability, we further make the following assumption.

**Assumption 4.** There exists a \( \mathbb{F} \)-progressively measurable non negative process \( \lambda \) such that \( \mathbb{P} \) The process \( \lambda \) is called the intensity of the random time \( \tau \).

The process \( \lambda \) can be interpreted as the conditional rate of arrival of the event at time \( t \geq \tau \), given all information available up to that time. This setup is similar to the one used in reduced-form models of default (see for example Duffie and Singleton (1999) or Lando (1997)). In the life-insurance literature, \( \lambda \) is usually known as a hazard rate, denoting the fact that, for a small interval of time \( \delta \), the conditional probability at time \( t \) that death occurs between \( t \) and \( t + \delta \), given survival up to \( t \), is approximately \( \lambda t \delta \), or

\[
\lambda_t = \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P} \left( t < \tau \leq t + \delta | \mathcal{F}_t \right) / \mathbb{P} \left( t < \tau | \mathcal{F}_t \right)
\]

It should be noted that if \( \lambda \) is deterministic, then \( \tau \) is independent of \( \mathbb{F} \). When the intensity is a constant \( \lambda \), \( \tau \) is simply the date of the first jump of a standard Poisson process. In other words, this setup is a natural generalisation of the familiar notion of an exponentially distributed random time.

Let us now give an example of a random time, which is not a stopping time of the \( \mathbb{F} \) filtration, while being not independent from \( \mathbb{F} \) and satisfying assumptions 3 and 4. One such uncertain time is the first hitting time of a stochastic barrier \( \Xi \).

\[
\tau = \inf \left\{ t : \int_0^t \lambda_s ds \geq \Xi \right\}
\]

where \( \lambda \) is any \( \mathbb{F} \)-progressively measurable non negative process and where \( \Xi \) is a random variable independent of the sigma-field \( \mathcal{F}_\infty \). For example, one may take \( \Xi \) to be an independent random variable exponentially distributed with parameter equal to 1. In this case, we have that

\[
\mathbb{P} \left[ \tau > t | \mathcal{F}_t \right] = \mathbb{P} \left[ \Xi > \int_0^t \lambda_s ds | \mathcal{F}_t \right] = \mathbb{P} \left[ \Xi > \int_0^t \lambda_s ds | \mathcal{F}_\infty \right] = \exp \left[ -\int_0^t \lambda_s ds \right]
\]

We also need one additional technical assumption about the random time.

**Assumption 5.** The random time \( \tau \) is finite almost surely, i.e., \( \mathbb{P}(\tau < \infty) = 1 \).

That assumption, along with assumption 4, implies that \( \exp \left[ -\int_0^\infty \lambda(s) ds \right] = 0 \).

Note that this assumption induces no real loss of generality since we could have obtained the same results by conditioning on the set \( \tau < \infty \).

1.3 Information Structure

In the context of an uncertain time-horizon, we model the information set available to the agents in such a way that it encompasses at any date \( t \) information about past values of asset prices \( \{\mathcal{F}_t\} \), and also information about whether the event of interest has occurred or not \( \{\mathcal{N}_t := \sigma(\tau \wedge t)\} \). The smallest filtration satisfying that property is known as the "progressive enlargement", denoted by \( \mathcal{G}_\tau \), of \( \mathbb{F} \) with respect to \( \tau \) (see for example Jeulin (1980) or Dellacherie (1972) for more details).

**Definition 1.** Jeulin (1980). The progressive enlargement of \( \mathbb{F} \) with respect to a random time \( \tau \), denoted by \( \mathcal{G}_\tau \), is the smallest filtration containing \( \mathbb{F} \) of which \( \tau \) is a stopping time. First introduce \( \mathcal{N}_t := \sigma(\tau \wedge t) \), the filtration generated by the family \( \tau \wedge t \), where \( \tau \wedge t \)

7 - There exists an alternative formulation of assumption 4 in terms of a point process perspective (see for example Dellacherie (1972)).
denotes $\inf (\tau, t)$. By definition, the filtration $\mathcal{G}$ is taken to be the smallest right continuous family of sigma-fields such that both $\mathcal{F}_t$ and $\mathcal{N}_t$ are in $\mathcal{G}_t$.

Note that if $\tau$ is taken to be a $\mathbb{F}$-stopping time, this enlargement is a trivial operation ($\mathcal{G} = \mathbb{F}$).

2. Dynamic Pricing with Uncertain Time-Horizon

Using the framework developed in Section 1, we may now address the valuation problem under consideration. The problem is to find the fair price at any date $t < \tau$ of a contingent claim with pay-off $\phi(S_\tau)$ at date $\tau$, where $\phi$ is a bounded function or such that $\phi(S_\tau)$ is with bounded quadratic variation. Since we have assumed that the $\mathbb{F}$-market is complete, any random variable measurable with respect to the filtration generated by asset prices is replicable using a suitable dynamic trading strategy. In particular, we have that $\phi(S_\tau)$ is replicable for all possible $t$. When the random time-horizon is not a stopping time of the filtration generated by asset prices, uncertainty over the timing of the cash-flow induces, however, some form of market incompleteness, which implies that there is no unique equivalent martingale measure (EMM). In what follows, we provide an explicit characterisation of the set of EMMs in the presence of an uncertain time-horizon.

2.1 Equivalent Martingale Measures

The following proposition provides an explicit characterisation of the set $\mathcal{E}$ of equivalent martingale measures, as well as the relationship between the intensity process under the original and under a new equivalent measure. A probability measure $\mathbb{Q}$ in $\mathcal{E}$, equivalent to $\mathbb{P}$, shall be regarded as a risk-neutral measure with respect to both asset price and timing risks. In other words, we are characterising the set of probabilities $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which discounted prices of cash-flows paid at the random date $\tau$ of the form $\phi(S_\tau)$, where $\phi$ is such that $\phi(S_\tau)$ is square integrable, are martingales.

Proposition 1 The set $\mathcal{E}$ of all possible EMMs is given by

$$\mathcal{E} = \left\{ \mathbb{Q}_H; \exists H_t \text{ s.t. } \left| \frac{d\mathbb{Q}_H}{d\mathbb{P}} \right|_{\mathcal{G}_t} = \xi_0(t) \times \xi_1(t); \mathbb{E}^\mathbb{P}(\xi_0(\tau) \times \xi_1(\tau)) = 1 \right\}$$

(3)

with $H_t$ being a (bounded) adapted process, and

$$\xi_1(t) = \exp \left( H_t 1_{\{t \leq \tau\}} - \int_{t}^{\tau} (e^{H_s} - 1) \lambda_s ds \right)$$

(4)

$$\xi_0(t) = \exp \left( - \int_{0}^{t} \beta_s dW_s - \frac{1}{2} \int_{0}^{t} \beta_s^2 ds \right)$$

(5)

and where $\beta_t$ has already been defined as $\sigma_t^{-1}(\mu_t - r_t)$. Furthermore, $\tau$ admits the intensity process

$$\tilde{\lambda}_t = e^{H_t} \lambda_t$$

(6)

under $\mathbb{Q}_H$, where $\lambda_t$ is the intensity process under the original measure $\mathbb{P}$. Moreover assumption 3 holds for all elements of $\mathcal{E}$.

Proof. See the Appendix. 9

The above proposition shows that a convenient multiplicative separation of asset price and timing risk-adjustments exists, a result essentially driven by assumption 3. On the one hand, the term $\xi_0$ in equation (5) solely affects the asset return process defined in equation (1) and has no impact on the intensity process of the random time. It provides a pure market risk adjustment; it

8 - The family $\mathcal{N}_t = \{\mathcal{N}_t; t \geq 0\}$ may not be a right-continuous filtration and the so-called "standard conditions" (see for example Karatzas and Shreve (1991)) may not be satisfied. For this reason, one typically needs to rather consider the right-regularisation of $\mathcal{N}_t$, $\mathcal{N}_t = \{\mathcal{N}_t; t \geq 0\}$ where $\mathcal{N}_t = \mathcal{F}_{t\wedge \tau}$, $\mathcal{N}_{t\wedge \tau}$.

9 - Changes of measure for discontinuous processes have been used for example in insurance literature by Aase (1999), Delbaen and Haezendonck (1989) and Sondermann (1991), and by Jarrow and Madan (1991) and Jarrow and Turnbull (1995) in finance literature. A classic mathematical reference is Brémaud (1981).
is the standard adjustment to the original probability of various price path scenarios performed by investors to account for aversion with respect to asset price risk. Hence, \( \beta \) is the traditional \textit{market price of market risk}. Within the context of our model, because there is one random perturbation and one traded asset, it is uniquely defined as \( \beta_t = \frac{\mu_t - \delta_t}{\sigma_t} \). On the other hand, the term \( \xi \) in equation (4) is the Radon-Nikodym derivative of a change of measure with respect to timing uncertainty. It affects the intensity process of the uncertain time but not the return process, and captures a market adjustment for timing risk. This allows us to introduce the following definition.

\textbf{Definition 2.} We define the \textit{market price for timing risk} as the logarithm of the ratio of the risk-adjusted intensity of \( \tau \) to the original intensity of \( \tau \), that is \( H_t = \ln \frac{\tilde{\lambda}_t}{\lambda_t} \).

Hence a zero market price for timing risk coincides with no adjustment for time-horizon probability distribution. Using the original intensity process \( \lambda \) for pricing purposes is equivalent to making the assumption of risk-neutrality with respect to timing risk. In this case, \( H_t := 0 \). A possible justification would be that timing risk may be diversified away; in all other cases, a risk-neutral intensity process \( \tilde{\lambda} \) should be used for pricing purposes.\(^{10}\)

\textbf{Remark 1.} For all \( H \), the restriction of \( Q_H \) on the subfield \( \mathcal{F}_t \) is equal to \( Q_0 \), the equivalent martingale measure with Radon-Nikodym derivative with respect to \( P \) is \( \xi_0 \). Note that \( Q_0 \) is also the equivalent martingale measure defined by \( H = 0 \), which corresponds to a zero risk premium associated with timing risk.\(^{11}\)

An equivalent way of formulating the message in (3) is to state the results in terms of the pricing kernel. It is well-known (see for example Duffie (2001)) that a stochastic discount factor (or pricing kernel) \( \pi \) is related to a given EMM \( Q_H \) by the following relationship

\[ \pi_t = \exp \left( -\int_0^t r_s ds \right) \mathbb{E}_t^P \left( \frac{dQ_H}{dP} \right) \]

Here, we have that, for \( t < \tau \)

\[ \pi_t 1_{\{t<\tau\}} = \exp \left( -\int_0^t r_s ds \right) \xi_0(t) \xi_1(t) 1_{\{t<\tau\}} \]

Using Itô’s lemma, one may compute explicitly the dynamics of the stochastic discount factor. We have that, for \( t < \tau \)

\[ \frac{d\pi_t}{\pi_t} = -r_t dt + \frac{d\xi_0(t)}{\xi_0(t)} + \frac{d\xi_1(t)}{\xi_1(t)} = -r_t dt - (e^{H_t} - 1) \lambda_t dt - \beta_t dW_t \]

From equation (7), we see that the impact of an uncertain time-horizon affects the pricing kernel through a suitable adjustment of the risk-free rate.

2.2 Pricing a Claim paying off at a Random Date

It is transparent from the above discussion that, while the absence of arbitrage opportunities implies that the set \( \mathcal{E} \) is not empty (see Harrison and Kreps (1979) and Harrison and Pliska (1981)), uniqueness, on the other hand, is not granted. Even when the asset markets are complete, as is the case here, and \( \xi_0 \) is uniquely defined through equation (5), there is an infinite number of possible equivalent martingale measures depending upon a choice for the market price of timing risk \( H \). This is because uncertainty about timing induces a specific form of market incompleteness in the general case when the random time is not a stopping time of the asset price filtration.

\[^{10}\] This is similar to option pricing in the presence of stochastic volatility. For example, in Hull and White (1987), stochastic volatility risk is assumed not to be rewarded. While a "price" is obtained under that assumption, no perfect hedging strategy is possible due to a market incompleteness induced by stochastic volatility. Another example is option pricing when the underlying asset follows a mixed diffusion-jump process. In Merton (1976), it is assumed that jump risk is non-systematic, and hence not rewarded in a CAPM framework. When jump risk can not be diversified away, however, one needs to derive the price for jump risk by some equilibrium argument (see for example Naik and Lee (1995), Ahn (1992) and Chang and Chang (1996)).

\[^{11}\] This follows from the orthogonality of \( W \) and \( M \) where \( M \) is defined by \( M_t = I_{\{t<\tau\}} - e^{-\int_0^t \beta_s ds} \lambda_t \) (see proof of proposition 1).
2.2.1 General Pricing Formula and Applications
The following proposition offers a general pricing formula for an asset with a pay-off occurring at a random date where we use some EMM in $\varepsilon$. In a next sub-section, we discuss several options to narrow down the number of admissible equivalent martingale measures.

**Proposition 2** Assume that $\Phi$ is such that $\mathbb{E}^\varepsilon[\Phi(\tau)^2] < \infty$. Given an EMM $\mathcal{Q}_H$ in $\varepsilon$, the price $J_t$ at date $t$ of an asset paying off a random cash-flow $\Phi(S_\tau)$ at a random time $\tau$, if prior to the maturity date of the contract $T$, is for $t < \tau$.

$$J_t = \mathbb{E}^{\mathcal{Q}_H} \left[ \exp \left( - \int_t^\tau r_s ds \right) \Phi(S_\tau) 1_{\{\tau < T\}} \right] G_t$$

$$= \mathbb{E}^{\mathcal{Q}_H} \left[ \int_t^T \exp \left( - \int_t^\tau (r_u + \hat{\lambda}_u) du \right) \hat{\lambda}_s \Phi(S_s) ds \right] F_t$$

(8)

where $\hat{\lambda}$ is the intensity process of $\tau$ under $\mathcal{Q}_H$.

**Proof.** This result is essentially obtained by using the law of iterated expectations and expressing the price as a risk-neutral average of risk-neutral expectations over all possible values $s$ for $\tau$. See the Appendix for a formal proof.

Using the Feynman-Kac formula, one may also, in a case with constant intensity, write an equivalent formulation of the above expectation in terms of the following partial differential equation (P.D.E)\(^{12}\)

$$\frac{\partial J_t}{\partial t} + \frac{1}{2} \sigma^2 J_t \frac{\partial^2 J_t}{\partial \tau^2} + r J_t \frac{\partial J_t}{\partial \tau} S_t - \left( \hat{\lambda} + r \right) J_t + \hat{\lambda} \Phi(S_t) = 0$$

(9)

subject to $J_T = \Phi(S_T)$.

As a simple illustration of the general pricing formula (8), we provide an explicit expression for the fair price, denoted by $I_t$, at date $t$ of a non-contingent insurance contract paying $1$ at date $\tau$, if $\tau$ occurs before $T$, the maturity date of the contract, i.e., we take in what follows $\Phi() = 1$ almost surely. The absence of arbitrage implies the existence of an EMM $\mathcal{Q}_H$ in $\varepsilon$ such that, for $t < \tau$, the price of the contract is given by

$$I_t = \mathbb{E}^{\mathcal{Q}_H} \left[ \exp \left( - \int_t^\tau r_s ds \right) 1_{\{\tau < T\}} \right] G_t$$

The following proposition provides a simple closed-form expression for that price in a case with constant intensity and constant interest rate.

**Proposition 3** For $t < \inf(\tau, T)$, the price of an asset paying off $1$ at date $\tau$, with intensity $\hat{\lambda}$ under some EMM $\mathcal{Q}$, is

$$I_t = \frac{\hat{\lambda}}{\hat{\lambda} + r} \left( 1 - \exp \left( - \left( \hat{\lambda} + r \right) (T - t) \right) \right)$$

(10)

and is equal to zero after that.\(^{13}\) If the contract has an infinite maturity ($T = \infty$), then the price is a constant $I = \frac{\hat{\lambda}}{\hat{\lambda} + r}$.

\(^{12}\) That equation will be independently derived in Section 3 by an explicit characterisation of dynamic hedging strategies for a cash-flow received at a random time (see equation (15)).

\(^{13}\) That $I_t$ is zero after $\tau$, even if $t < \tau$, is because we assume that the asset pays off only once in its lifetime.
Proof. Immediate from equation (8) applied to a pay-off of the form \(1_{\{s < T\}}\).

We note that \(I_t = 1\) if \(r = 0\) in case of an infinite maturity contract, as it should. Indeed, the assumption of a zero interest rate implies no time-value of money, and the assumption of an infinite maturity implies that we have shifted from a "if the event occurs" perspective to a "when the event occurs" perspective. In that context, the present value of $1 received for certain at some future date when the discount factor is identically equal to one must be $1. On the other hand, \(I_t = 0\) if \(\hat{\lambda} = 0\), which is consistent since in that case there is a 100% probability that the asset will never pay-off anything.

As another illustration, we consider a catastrophe home insurance contract. Buyers of catastrophe insurance usually expect a coverage related to the value of the house upon the date \(\tau\) of a catastrophic event such as an earthquake. Hence, they buy a contract which pays off, at date \(\tau\), some function \(\Phi\) of the value of the house \(E\), at the date of the event, if that date is prior to the maturity date of the contract \(T\). Typically, \(\Phi\) may capture a maximum amount \(K\) of coverage, in which case the pay-off \(E_\tau = \min(E_\tau, K) 1_{\{\tau < T\}} = \{\min(E_\tau - K, 0) + K\} 1_{\{\tau < T\}}\). This is a limited coverage contract. We note that the pay-off of a limited coverage contract is identical to a certain pay-off equal to the strike price \(K\) minus the pay-off of a European put with random maturity date \(\tau\) - \(\min(K - E_\tau, 0)\). The price for that contract is given by the following proposition.

**Proposition 4** We assume that the price of the real estate asset, denoted by \(E\), follows a geometric Brownian motion

\[
\frac{dE_t}{E_t} = \mu dt + \sigma dW_t
\]

The fair price of an insurance contract with limited coverage \(K\) on a catastrophic event occurring at date \(\tau\), with a constant intensity process under \(\mathcal{Q}_H\), denoted by \(\widehat{\lambda}\) is given by

\[
J_0 = K I_0 - P_\tau(0)
\]  

where \(I_0\) is given in equation (10), and \(P_\tau(0)\) is given in equation (12)

\[
P_\tau(0) = \left(\frac{E_0}{K}\right)^{\gamma - \varepsilon} (qKR - \widehat{q}K)
\]  

where \(\gamma := \frac{1}{2} - \frac{R}{\sigma^2}; R := \frac{1}{1 + \frac{\gamma}{\sigma^2}}; S := \frac{1}{\lambda}; \varepsilon := \sqrt{\gamma^2 + \frac{2}{\lambda R^2}}; q := 1 - \frac{\varepsilon - \gamma}{2\varepsilon}; \) and \(\widehat{q} := 1 - \frac{\varepsilon - \gamma + 1}{2\varepsilon}\).

Proof. See the Appendix.

It is useful to explicitly characterise the limits of insurance contract prices for extreme values of the risk-neutral intensity. For simplicity, we discuss a typical case with \(K = E_0\). When \(\widehat{\lambda} = 1\), we get \(I_0 = 1\), as it should since an infinite intensity means that a catastrophe is about to occur with (risk-neutral) probability equal to one. To compute \(P_\tau(0)\) in that case, just note that \(\widehat{\lambda} = 1\) implies \(S = \infty, R = 1, \varepsilon = \infty\) and \(q = \widehat{q} = \frac{1}{2}\). Then, using \(P_\tau(0) = qKR - \widehat{q}K\), we get \(P_\tau(0) = 0\), as it should. Indeed, since a catastrophe is about to occur, the option has no time value, its price is equal to the intrinsic value, that is zero since the option is at the money \((K = E_0)\). Finally, \(J_0 = K\) in that case. On the other hand, if \(\widehat{\lambda} = 0\), that is if the (risk-neutral) probability of catastrophe occurring is zero, we get \(I_0 = 0\), as it should. To compute \(P_\tau(0)\) in that case, we note that \(\widehat{\lambda} = 0\) implies \(S = \infty, R = 0,\) and \(RS \sim \frac{1}{\varepsilon^2}\). Then \(\varepsilon = \sqrt{\gamma^2 + \frac{2}{\lambda}}\) or equivalently, after some algebra and using the definition of \(\gamma, \varepsilon = \frac{1}{2} + \frac{R}{\sigma^2}\). In that case, we have \(\widehat{q} = 1 - \frac{\varepsilon - \gamma + 1}{2\varepsilon} = 0\). Finally, we get \(P_\tau(0) = -\widehat{q}K = 0\) and \(J_0 = 0\). This is an intuitive result: if the (risk-neutral) probability of a catastrophe happening is zero, then insurance contracts never pay-off, and their value should be equal to zero.

2.2.2 Arbitrage Bounds and Imperfect Hedging

We now attempt to narrow down the number of admissible martingale measures by arbitrage arguments only.\(^{14}\) We first consider the concept of (almost surely) perfect replication, and then move on to several concepts of imperfect hedging.
Perfect Replication and Arbitrage Bounds In this section we discuss arbitrage bounds. In an incomplete markets situation, an infinite number of prices are consistent with no arbitrage.

As is well known, the absence of arbitrage imposes that the value of a contingent claim written on the underlying asset falls in an interval whose endpoints corresponds to the so-called hedging prices.

The upper bound is the upper hedging price which corresponds to the minimum initial endowment necessary to cover a short position in the claim. On the other hand, the lower bound corresponds to the lower hedging price which is the largest initial debt that can be contracted along with a long position in the claim without going bankrupt.

In this context, is desirable to know what are the price bounds consistent with the absence of arbitrage for a random pay-off paid at a random date. The following proposition shows that such bounds are actually too wide to be of any practical use.

Proposition 5 i) Assume that \( \Phi \) is bounded. The lower bound on the price at date \( t \) of an asset paying off a random cash-flow \( \Phi(S_\tau) \) at a random time \( \tau \), if prior to the maturity date of the contract \( T \), is equal to zero. The upper bound on the price at date \( t \) of an asset paying off a random cash-flow \( \Phi(S_\tau) \) at a random time \( \tau \) is equal to \( \sup_x \Phi(x) \):

ii) Assume that \( \Phi \) satisfied \( 0 \leq \Phi(x) \leq x \). The lower bound on the price at date \( t \) of an asset paying off a random cash-flow \( \Phi(S_\tau) \) at a random time \( \tau \), if prior to the maturity date of the contract \( T \), is equal to zero. The upper bound on the price at date \( t \) of an asset paying off a random cash-flow \( \Phi(S_t) \) at a random time \( \tau \) is equal to \( S_t \).

Proof. See the Appendix.

The intuition behind these results is rather straightforward. Let us for example focus first on the case of a function \( \Phi \) such that \( 0 \leq \Phi(x) \leq x \), a case which encompasses for example the standard European call pay-off function. In this case, the lower bound corresponds to a random time with zero intensity, which means that the payment date is almost sure not to be taking place in a finite amount of time; on the other hand, the upper bound corresponds to a random time with infinite intensity, which means that the payment date is date \( t \) with probability 1, and therefore the price is equal to \( S_t \), the maximum of the pay-off at date \( t \). This result is actually an extension to a random maturity setup of the result proved by Soner, Shreve and Cvitanic (1995), who show that initially buying a share of the underlying stock is the cheapest dominating policy for option replication in the presence of transaction costs.

Given that arbitrage bounds are trivially large, further justification for a specific choice of an EMM among all possible elements of \( \varepsilon \) is needed. Different routes haven been followed in the literature on asset pricing theory in the presence of incomplete markets.

Imperfect Hedging and the Minimal Martingale Measure In the presence of an uncertain time-horizon, perfect replication of a random pay-off is not possible in general because markets are incomplete, and one may instead focus on imperfect replication strategies. The mean-variance optimal hedging strategy, corresponding to the so-called minimal martingale measure, is one particular such strategy that has been discussed in the literature. The aim of the mean-variance optimal hedging strategy introduced by Föllmer and Schweizer (1991) is to minimise the variance between the random pay-off and the terminal wealth generated from a self-financing strategy.

In financial terms, it provides an approximation of the contingent claim by means of a self-financing trading strategy with minimal global risk, where risk is measured as the tracking error (variance of the replication error). More formally, we introduce the quadratic risk \( D^H \) of a contingent claim \( \Phi(S_t) \), defined as

\[
D^H = \mathbb{E}^\mathbb{Q}_H[(Y_T^H - Y_t^H)^2/G_t]
\]
where \((Y_t^H, t \geq 0)\) is a \(\mathcal{Q}_H\)-square martingale, with expectation equal to zero, orthogonal to the process \(RS\) such that

\[
\mathbb{E}^{\mathcal{Q}_H}[\Phi(S_T)1_{\{\tau < T\}}R_T/\mathcal{G}_t] = \mathbb{E}^{\mathcal{Q}_H}[\Phi(S_T)1_{\{\tau < T\}}R_T] + \int_0^t \xi_s d(RS)_s + Y_t^H
\]

where \(R_t := \exp\left(-\int_0^t r_s ds\right)\).

We therefore have the following definition.

**Definition 3.** An EMM \(\mathcal{Q}_H\) in \(\mathcal{E}\) is called minimal martingale measure if

i) \(\mathcal{Q}_H\) is equivalent to \(\mathbb{P}\).

ii) \(\mathcal{Q}_H = \mathbb{P}\) on \(\mathcal{G}_0\):

iii) Every \(\mathbb{P} - \mathcal{G}\)-square martingale orthogonal to \(W\) under \(\mathbb{P}\) is a \(\mathcal{Q}_H\)-martingale.

**Proposition 6** The minimal martingale measure is \(\mathcal{Q}_0\), i.e. the equivalent martingale measure defined by \(H = 0\), which corresponds to a zero risk premium associate with timing risk.

**Proof.** See the Appendix.

The interpretation of the result is that an agent who wishes to (imperfectly) hedge a random pay-off received at a random time-horizon prices securities as if he/she were risk-neutral with respect to timing uncertainty. For example, if we get back to the case of the contract introduced in proposition 3, the price of an asset paying off \$1 at date \(\tau\) is

\[
I_t = \frac{\lambda}{\lambda + r} (1 - \exp(- (\lambda + r)(T - t)))
\]

where \(\lambda\) is the original intensity of \(\tau\) (under \(\mathbb{P}\)).

**Imperfect Hedging and the Minimum Entropy Measure** Another natural choice for an EMM in the set \(\mathcal{E}\) is the (unique) EMM minimising the relative entropy to the original measure \(\mathbb{P}\), which has the interpretation of a measure of mispricing error (see for example Fritelli (2000)). More formally, we introduce the following definition.

**Definition 4.** Let \(\mathcal{Q}_H\) be an EMM in \(\mathcal{E}\). We define the relative entropy of \(\mathcal{Q}_H\) with respect to \(\mathbb{P}\), denoted by \(I(\mathcal{Q}_H, \mathbb{P})\), by

\[
I(\mathcal{Q}_H, \mathbb{P}) = \mathbb{E}^\mathbb{P}(\frac{d\mathcal{Q}_H}{d\mathbb{P}} \ln \frac{d\mathcal{Q}_H}{d\mathbb{P}})
\]

In the following proposition, we characterise the minimum entropy measure in an economy with uncertain time-horizon.

**Proposition 7** The minimum entropy measure is also \(\mathcal{Q}_0\), i.e. the equivalent martingale measure defined by \(H = 0\), which corresponds to a zero risk premium associated with timing risk.

**Proof.** See the Appendix.

We find again that the optimal pricing rule is one of a risk-neutral agent with respect to timing uncertainty. In other words, one may simply use the original intensity process in the pricing formulas.

**Completing the Markets** An alternative solution for narrowing down to one the number of possible EMMs consists in using market prices of a redundant security to perform relative pricing and dynamic hedging. This is the route we take in part of Section 3. Consider for example the case of the asset paying \$1 at date \(\tau\). If this contract is dynamically traded, one could in principle
extract from its price it a value for the risk-neutral intensity \( \hat{\lambda} \), and use it to price any other redundant asset. The intensity \( \hat{\lambda} \) may be regarded as an implied risk-neutral intensity, that is the value for \( \hat{\lambda} \) which reconciles market prices to the pricing formula (10). By differentiating equation (10), we get

\[
dI_t = -\hat{\lambda}e^{-(\hat{\lambda} + r)(T-t)} dt = -\hat{\lambda} dt + I_t \left( \hat{\lambda} + r \right) dt
\]

so that the term \( \hat{\lambda} \) is given by

\[
\hat{\lambda} = \frac{r I_t - dI_t}{1 - I_t}
\]

or simply \( \hat{\lambda} = \frac{r I_t}{1 - I_t} \) in a case with infinite maturity. Note that a choice for \( \hat{\lambda} \) implies a unique specification of an EMM among all possible elements of \( \varepsilon \); the markets have been completed by the introduction of the generalised risk-free asset. In Section 3, we use this contract to derive an explicit dynamic hedging strategy for a general contract with random maturity, and find an equation identical to (13).

3. Dynamic Hedging with Uncertain Time-Horizon

We now discuss the problem of hedging an asset which pays off at a random time. More specifically, we consider the problem of hedging a general contingent claim with price \( J_t = J(t, S_t) \) at date \( t \), and payoff \( \Phi(S_\tau) \) at date \( \tau \). The intuition behind the results we obtain in this section is straightforward. Asset price risk may be hedged by using a dynamic trading strategy in the underlying asset. Timing risk, on the other hand, may only be hedged by a suitable dynamic trading strategy involving some redundant asset, of which the pay-off is also contingent upon the timing of the event of interest. We shall provide here both a rigorous formulation of the hedging problem in terms of an extension of the standard martingale representation theorem and also an heuristic (but explicit) formulation in terms of a partial differential equation (P.D.E.).

3.1 Martingale Formulation

In a standard case with certain time to maturity, the martingale representation theorem allows one to characterise the set of replicable contingent claims. The following result shows that an extension of this theorem holds in the more general case of a random time horizon.

**Theorem 1** Let \( X_\tau := \Phi(S_\tau) \), where \( \Phi \) is a bounded function. We introduce the process

\[
\mu_t^X := \exp \left( \int_0^t \tilde{\lambda}_u du \right) J_t - \exp \left( \int_0^t \tilde{\lambda}_u du \right) \int_0^t X_s \exp \left( -\int_0^s (r_u + \tilde{\lambda}_u) du \right) \tilde{\lambda}_u ds
\]

for all \( t \in [0, \tau] \) and some measure \( \mathbb{Q}_H \) in \( \varepsilon \), where \( J_t \) is the martingale defined, for all \( t \in [0, \tau] \),

Then the martingale \( M_t^X := \mathbb{E}_t^{\mathbb{Q}_H} \left[ \exp \left( -\int_0^\tau r_s ds \right) X_\tau \right] \) admits the following decomposition

\[
M_t^X = J_0 + \int_0^{\tau \wedge \tau} \exp \left( \int_0^s \tilde{\lambda}_u du \right) dJ_s + (X_\tau - \mu_\tau^X) 1_{\{\tau \leq t\}} - \int_0^{\tau \wedge \tau} \left( X_s - \mu_s^X \right) \tilde{\lambda}_s ds
\]

**Proof.** See the Appendix.

This theorem provides formal content to a natural financial intuition, that we recover in the P.D.E. formulation below. The value \( J_0 \) is the "price", subject to the characterisation of \( \mathbb{Q}_H \) of the contingent claim \( X_\tau = \Phi(S_\tau) \) (see equation (8)). The first term on the right handside of equation (14) is the continuous martingale part of the decomposition. From a financial standpoint, it is what one can expect to hedge and replicate using mere dynamic trading in the basic risky asset.

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15 - The martingale representation theorem roughly states that every squared integrable martingale of the filtration generated by a standard Brownian motion may be represented as a stochastic integral with respect to that Brownian motion (see for example Karatzas and Shreve (1991)).

16 - We refer the reader to Blanchet-Scalliet and Jeanblanc (2004) for a more general version of this theorem in a context where the K-assumption (assumption 3) does not apply.
If no other hedging asset is used, that dynamic hedging strategy is imperfect, and one is left with a tracking error given by $X_t - \mu^e_t$.

3.2 P.D.E. Formulation
We first discuss the question of hedging solely asset price risk, and then hedging both asset price and timing risks.

3.2.1 Hedging Asset Price Risk
We assume interest rates and the intensity process are deterministic, or adapted and functions of $t$ and $S_t$ so that the basic market is complete, that is before the introduction of timing risk. The analysis could easily be generalised to account for interest rates and intensity risks, provided that more hedging assets can be used to hedge these additional risks (see discussion below). We consider some hedging portfolio invested in the risky asset and the contract with payoff $\Phi(S_t^\tau)$.

We denote by $P_t = J(t, S_t) + \phi_t S_t$ the value of that portfolio at date $t$, where the quantity $\phi_t$ is to be optimally derived. Using the assumption of a self-financing portfolio up to date $\tau$, we obtain that, over a small interval of time $dt$, the change in value of the hedging portfolio is given by $dP_t = dJ_t(t, S_t) + \phi_t dS_t$. Note that $P_t$ is not a continuous process; a jump takes place in the portfolio value at date $\tau$.

Equation (14) suggests that a heuristic derivation of the optimal hedging strategy is as follows.

If the event triggering $\tau$ does not occur at time $t + dt$ (probability $1 - \lambda dt$), then, for $t < \tau$

$$dP_t = \left( \frac{\partial J_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 J_t}{\partial x^2} \right) dt + \frac{\partial J_t}{\partial x} dS_t + \phi_t dS_t$$

One may choose $\phi_t$ so as to eliminate asset price risk. This is done by taking $\phi_t = -\frac{\partial J_t}{\partial x}$, which is essentially a neutral delta requirement.

If the event driving $\tau$ does occur (probability $\lambda dt$), then one suddenly loses the time value of the contract, and $dP_t = \Phi(S_t) - J_t$, from the standpoint of an investor who is long the contract, the other terms are negligible.

Taking (iterated) expectations both with respect to market uncertainty and timing uncertainty, we require that the return on the portfolio be equal to the risk-free interest rate $\mathbb{E}_t (dP_t) = r P_t dt$. Since timing risk is not hedged, one should not however take this expectation under the original probability $\mathbb{P}$, but under an EMM $\mathbb{Q}_H$, unless there is a specific reason to assume that the investor is risk-neutral with respect to time-horizon risk. In other words, one should use some $\lambda$ and not $\lambda$ in the expectation. Keeping only higher order terms, we obtain

$$\left( \frac{\partial J_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 J_t}{\partial x^2} \right) + \lambda \left[ \Phi(S_t) - J_t \right] = r \left( J_t - \frac{\partial J_t}{\partial x} S_t \right)$$

which we re-arrange into the following P.D.E.

$$\frac{\partial J_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 J_t}{\partial x^2} + r \frac{\partial J_t}{\partial x} S_t - (\lambda + r) J_t + \lambda \Phi(S_t) = 0$$

subject to proper boundary conditions. If the contract has an infinite maturity, then we are considering an infinite time-horizon problem and the term $\frac{\partial J_t}{\partial t}$ disappears, so that we are left with a stationary partial differential equation. Note that this P.D.E. is identical to the one obtained in Section 2 as an application of the Feynman-Kac theorem (equation (9)). For the sake of illustration, let us consider the problem of hedging some European call with a maturity 3 months or 6 months, with risk-adjusted probability .25 and .75. The dynamic hedging strategy described here consists
in holding at each date \( t \) a quantity of underlying asset equal to \( .25 \delta_{t,3} + .75 \delta_{t,6} \), where \( \delta_{t,T} \) is the standard Black-Scholes delta at date \( t \) for an option with maturity \( T \). It should be noted that this hedge is not perfect, unless one introduces some asset to complete the market, as is now discussed.

### 3.2.2 Hedging both Asset Price and Timing Risks

As argued above, dynamic trading in the risky asset does not allow one to hedge the jump component due to realisation of the uncertain time \( \tau \). Completing the market may only be achieved by introducing some asset with a pay-off contingent upon the same date \( \tau \). For concreteness, we use the generalised risk-free asset with price \( I_t \) as an hedging instrument. That asset pays off \$1 at date \( \tau \), whatever the state of the world with respect to asset prices at that date. It is the generalised risk-free asset in an economy with uncertain time-horizon \( \tau \). We consider some hedging portfolio invested in the risky asset and that generalised risk-free asset \( P_t = J(t, S_t) + \phi_1 S_t + \phi_2 I_t \). Using the requirement of a self-financing portfolio up to date \( \tau \), we obtain \( dP_t = dJ(t, S_t) + \phi_1 dS_t + \phi_2 dI_t \). We then repeat the same heuristic analysis as above.

If the event does not occur (probability \( 1 - \lambda dt \)), then

\[
dP_t = \left( \frac{\partial J_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 J_t}{\partial x^2} \right) dt + \frac{\partial J_t}{\partial x} dS_t + \phi_1 dS_t + \phi_2 dI_t
\]

If the event occurs (probability \( \lambda dt \)), then one suddenly loses the time value of the contract to be hedged while the generalised risk-free asset pays off \$1. Hence \( dP_t = (\Phi(S_t) - J_t) + \phi_2 (1-I_t) \), the other terms being negligible.

Here, we may choose \( \phi_1 \) and \( \phi_2 \) so as to eliminate both asset price and timing risks. This can be done by taking \( \phi_1 = -\frac{\partial \Phi}{\partial x} \) and \( \phi_2 = -\frac{\Phi(S_t) - J_t}{1-I_t} \). Finally, taking expectations, we write \( \mathbb{E}_t(dP_t) = rP_t dt \).

Here, all risks, including timing risk, are being hedged, so that taking expectations under the original probability \( \mathbb{P} \) is now valid. Keeping only higher order terms, we obtain the following P.D.E.

\[
\frac{\partial J_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 J_t}{\partial x^2} + r \frac{\partial J_t}{\partial x} S_t + \Phi(S_t) \left( \frac{dt}{\partial t} + r I_t \right) - J_t \left( \frac{dt}{\partial t} - r I_t \right) = 0
\]

subject to proper boundary conditions. This equation is similar to (15), except that we now have the term \( \frac{dt}{\partial t} \bigg|_{1-I_t} \) where we had \( \hat{\lambda} \) before. From this, we obtain that the risk-neutral intensity process is \( \hat{\lambda} = \frac{dt}{\partial t} \bigg|_{1-I_t} \). To check the consistency of the analysis, note that one gets the same result by extracting an implied risk-neutral intensity from the price of the generalised risk-free asset (see (13)).

Hence, when the intensity is a constant, one may hedge the jump component induced by the presence of timing risk by introducing one redundant asset to complete the market. If the intensity is stochastic, however, there are three risks involved: asset price risk; risk of the event happening; and risk of changes in the event probability. In that case, one would need one more asset to achieve perfect hedging.

## 4. Conclusion

The normative side of asset pricing theory aims at providing decision making rules for agents in the market. As such, it lies in the realm of substantive rationality: given the information available, agents always have the ability to compute the optimal solution to the problem they face. In this context, a major difficulty is to model the information structure faced by the agent, which is contingent on the nature of the problem to be solved. Two extreme cases may be considered, pure risk and pure uncertainty (see Knight (1921)). Pure risk is a situation where events are not certain, but will occur according to probability distributions which are known and stationary, or which evolution is known with certainty. These probabilities are the true probabilities. Pure
uncertainty, on the other hand, is a situation where agents have absolutely no knowledge about these probabilities and have no means to get information on them. Both settings entail a very strong degree of simplification, and real investment situations lie somewhere in between these two extremes. The challenge for an economist is then to find a way to model, in a simple but cogent way, a relevant degree of uncertainty with respect to the nature of the problem faced.

By relaxing the assumption that an agent always knows with certainty the exact timing of a cash-flow, and by allowing these probability to evolve in time in a stochastic way, this paper may be viewed as an attempt to cover some of the open territory between pure risk and pure uncertainty. A serious complication is that the problem involves two sources of uncertainty, one stemming from the randomness of asset prices, the other from the randomness of time. A further complication is that these two sources of uncertainty are generally not independent. Our contribution is to show that these two complications may be conveniently addressed within a unified and tractable framework. In particular, we provide an explicit arbitrage characterisation of the set of equivalent martingales measure in an economy with uncertain time-horizon (equations (3) to (5)), as well as necessary and sufficient conditions for a convenient separation to hold between adjustments for market risk and timing risk. Building on this result, we discuss in some detail arbitrage valuation and replication methods for a non-redundant asset with an uncertain maturity date.

A. Appendix

A.1 Proof of Proposition 1

Here may be found a proof for proposition 1. Let us first introduce the $\mathcal{G}$–compound martingale of the process $N_t := 1\{\tau \leq t\}$

$$M_t = N_t - \int_0^{\tau \wedge t} \lambda_s ds$$

The process $M$ is a martingale under the probability $\mathbb{P}$. Moreover it is well-known (Dellacherie and Meyer (1978)) that in our context assumption 3 is equivalent to the following statement: every $\mathbb{F}$–square martingale is a $\mathcal{G}$–square martingale. In particular $W$ is a $\mathcal{G}$–Brownian motion.

Let $\mathcal{Q}$ be an EMM. We denote by $K$ its Radon-Nikodym density with respect to $\mathbb{P}$: $K_t = \frac{d\mathcal{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$. The process $K$ is a $\mathcal{G}$–square martingale. As assumption (3) holds under $\mathbb{P}$, one can apply a representation theorem from Kusuoka (1999) for the martingale $K$. We first recall this result.

Theorem 2 (Kusuoka (1999)) If every $\mathbb{F}$–square martingale is a $\mathcal{G}$–square martingale, then for any $\mathcal{G}$–square martingale $\{Y_t\}$; there are $\mathcal{G}$–predictable processes $f: [0, \infty[ \times \rightarrow \mathbb{R}$ and $g: [0, \infty[ \times \rightarrow \mathbb{R}$ such that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^\infty |f_t|^2 dt\right] < +\infty \text{ and } \mathbb{E}^{\mathbb{P}}\left[\int_0^\infty |g_t|^2 \lambda_t dt\right] < +\infty$$

and

$$Y_t = Y_0 + \int_0^t f_s dW_s + \int_0^t g_s dM_s, \ t \in [0,T].$$

Noting that $K$ is positive, we obtain that there are two process $\psi$ and $\varphi$ $\mathcal{G}$–predictable processes such that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^\infty |K_t \psi_t|^2 dt\right] < +\infty \text{ and } \mathbb{E}^{\mathbb{P}}\left[\int_0^\infty |K_t \varphi_t|^2 \lambda_t dt\right] < +\infty$$

and

$$dK_t = K_t(\psi_t dW_t + \varphi_t dM_t)$$
Since $K$ is strictly positive, one has that: $-1 < \varphi_t < \infty$. We then apply Itô's formula to the process $RSK$, where $R$ is defined by

$$R_t := \exp \left( -\int_0^t r_s ds \right),$$

to obtain

$$d(RSK)_t = (RSK)_t \left( (\mu_t - r_t - \sigma_t \psi_t) dt + \sigma_t dW_t \right) + (RSK)_{t-} \left( -\psi_t dW_t + \varphi_t dM_t \right)$$

Since $RSK$ is a $\mathbb{P}$ martingale, then $\bar{A} = \beta$. Taking $H_t = \ln(1+\varphi_t)$; the result follows from orthogonality of $W$ and $M$:

Assumptions (1) and (2) ensure that $\beta$ is deterministic and therefore assumption (3) holds under $\mathbb{Q}$ (see Blanchet-Scalliet (2001) for more details on sufficient conditions for the stability of assumption 3 under a change of measure).

Moreover, assumptions (1) and (2) imply that $\beta$ is bounded, which ensures that the standard integrability condition $\mathbb{E}^\mathbb{P} \xi_0(t) = 1$ is satisfied for all $t \in [0, \infty[$. Moreover, conditions (16) imply that the integrability condition $\mathbb{E}^\mathbb{P} (\xi_1(t)) = 1$ is also satisfied for all $t \in [0, \infty[$ (see also theorem 11, Section VI in Brémaud (1981) for more details).

A.2 Proof of Proposition 2

We now provide a proof for the pricing formula (8). We assume that $Y$ is $\mathcal{F}_t-$adapted or bounded. For $t < \tau$, we have that (Bayes’ rule)

$$\mathbb{E}^\mathbb{Q}_H \left[ Y_t \left| \mathcal{G}_t \right. \right] 1_{\{t < \tau\}} = \frac{\mathbb{E}^\mathbb{Q}_H \left[ Y_t 1_{\{t < \tau\}} \left| \mathcal{F}_t \right. \right]}{\mathbb{E}^\mathbb{Q}_H \left[ 1_{\{t < \tau\}} \left| \mathcal{F}_t \right. \right]} 1_{\{t < \tau\}}$$

First recall that (because assumption 3 holds under $\mathbb{Q}_H$)

$$\mathbb{Q}_H[\tau > t| \mathcal{F}_\infty] = \exp \left( -\int_0^t \hat{\lambda}_s ds \right) := \exp \left( -\hat{\lambda}_t \right)$$

and, introducing the following conditional density function

$$g(t) = \frac{\partial}{\partial s} \mathbb{Q}_H[\tau \leq s| \mathcal{F}_\infty] \bigg|_{s=t}$$

we also have

$$g(t) = \hat{\lambda}_t \exp \left( -\int_0^t \hat{\lambda}_s ds \right) := \hat{\lambda}_t \exp \left( -\hat{\lambda}_t \right)$$

Then, we get (law of iterated expectations)

$$\mathbb{E}^\mathbb{Q}_H \left( Y_t 1_{\{t < \tau\}} \left| \mathcal{F}_t \right. \right) 1_{\{t < \tau\}} = \mathbb{E}^\mathbb{Q}_H \left( \mathbb{E}^\mathbb{Q}_H \left( Y_t 1_{\{t < \tau\}} \left| \mathcal{F}_\infty \right. \right) \left| \mathcal{F}_t \right. \right) 1_{\{t < \tau\}}$$

$$= \mathbb{E}^\mathbb{Q}_H \left( \int_t^\tau \exp \left( -\hat{\lambda}_s \right) \hat{\lambda}_s Y_s ds \left| \mathcal{F}_t \right. \right) 1_{\{t < \tau\}}$$

17 - Note also that under our maintained assumption $\mathbb{Q}_H[\tau > \tau_0] = \mathbb{Q}_H[\tau > \tau_0]$
Finally

\[ E^Q \left[ Y_t \mid \mathcal{G}_t \right] 1_{\{t < \tau\}} = \frac{E^Q \left[ Y_t 1_{\{t < \tau\}} \mid \mathcal{F}_t \right]}{E^Q \left[ 1_{\{t < \tau\}} \mid \mathcal{F}_t \right]} 1_{\{t < \tau\}} = \frac{\int_t^T \exp \left( -\hat{\lambda}_s \right) \hat{\lambda}_s Y_s ds \mid \mathcal{F}_t}{\exp \left( -\hat{\lambda}_t \right)} 1_{\{t < \tau\}} \]

\[ = \frac{\int_t^T \exp \left( -\hat{\lambda}_s + \hat{\lambda}_t \right) \hat{\lambda}_s Y_s ds \mid \mathcal{F}_t}{\exp \left( -\hat{\lambda}_t \right)} 1_{\{t < \tau\}} \]

\[ = \frac{\int_t^T \exp \left( -\int_t^s \hat{\lambda}_u du \right) \hat{\lambda}_s Y_s ds \mid \mathcal{F}_t}{\exp \left( -\int_t^T \hat{\lambda}_u du \right)} 1_{\{t < \tau\}} \]

Taking \( Y_t = \exp \left( -\int_t^T r_s ds \right) \Phi \left( S_t \right) 1_{\{t < \tau\}} \) concludes the proof.

A.3 Proof of Proposition 4

In the absence of arbitrage, we first note that the fair price of the contract is

\[ J_t = \mathbb{E}^Q \left[ \int_t^T \exp \left( -\int_t^s \left( r_u + \hat{\lambda}_u \right) du \right) \hat{\lambda}_s \left[ \min \left( E_s - K, 0 \right) + K \right] ds \mid \mathcal{F}_t \right] \]

\[ = \mathbb{E}^Q \left[ \int_t^T \exp \left( -\int_t^s \left( r_u + \hat{\lambda}_u \right) du \right) \hat{\lambda}_s \min \left( E_s - K, 0 \right) ds \mid \mathcal{F}_t \right] + K I_t \]

If we consider for simplicity deterministic risk-neutral intensity and interest rate processes, we get, using Fubini’s theorem for positive functions

\[ J_t = \int_t^T \exp \left( -\int_t^s \hat{\lambda}_u du \right) \hat{\lambda}_s \mathbb{E}^Q \left[ \exp \left( -\int_t^s r_u du \right) \min \left( E_s - K, 0 \right) \mid \mathcal{F}_t \right] ds + K I_t \]

or

\[ J_t = K I_t - \int_t^T \exp \left( -\int_t^s \hat{\lambda}_u du \right) \hat{\lambda}_s P \left( E_t, K, \sigma, r, s - t \right) ds \]

where \( P \left( E, K, \sigma, r, s - t \right) \) is the standard Black and Scholes (1973) price at date \( t \) for an European put option on \( E \) with a strike price \( K \) and a maturity date \( s \). In other words, \( P \left( r \right) = J_t - K I_t \), can be written as an average of Black-Scholes put prices for all possible maturity dates ranging from \( t \) to \( T \). A similar expression has been obtained by Carr (1998), where the author introduces a new method for pricing American options by fictitiously allowing the maturity date of the contract to be random, and then letting the variance of the random time go to zero. In mathematical terms, \( P \left( r \right) \) may also be regarded as the Laplace-Carson transform of a fixed maturity European call. In the case with infinite maturity \( T \), by taking the Laplace-Carson transform of both sides of the standard Black-Scholes PDE, Carr (1998) obtains the following closed-form solution for a European call with random maturity

\[ C \left( 0 \right) = \left( \frac{E_0}{K} \right)^{\gamma + \epsilon} \left( \tilde{p} K - p K R \right) \]

where \( \gamma := \frac{1}{2} - \frac{\sigma^2}{2} ; R := \frac{1}{1 + r S} ; S := \frac{1}{\lambda} ; \epsilon := \frac{\gamma}{2} + \frac{1}{2 \gamma} ; \lambda_R \). Using a generalised call-put parity for European options paying off at random times, the fair price of the European put with random maturity is\(^{18}\)

\[ P \left( 0 \right) = \left( \frac{E_0}{K} \right)^{\gamma - \epsilon} \left( q K R - \tilde{q} K \right) \]

\(^{18}\) Call-put parity holds provided that there is a forward contract which matures at the same uncertain time as the option (see footnote 6 in Carr [1997]).
where \( q := 1 - p \) and \( \tilde{q} := 1 - \tilde{p} \). Taking \( J_t = -P_t(t) = + K_t \) concludes the proof.

A.4 Proof of Proposition 5

i) Since the set of EMM is convex, the set of prices consistent with no arbitrage is an interval. More precisely, one has

\[
0 \leq \inf_{\mathbb{H}} \mathbb{E}^{\mathbb{Q}_H} [R_t \Phi(S_t) 1_{\{\tau < T\}} / \mathcal{G}_t] \leq \mathbb{E}^{\mathbb{Q}_H} [R_t \Phi(S_t) 1_{\{\tau < T\}} / \mathcal{G}_t] \leq \sup_{\mathbb{H}} \mathbb{E}^{\mathbb{Q}_H} [R_t \Phi(S_t) 1_{\{\tau < T\}} / \mathcal{G}_t]
\]

Since we have

\[
1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_H} [R_t \Phi(S_t) 1_{\{\tau < T\}} / \mathcal{G}_t] = 1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_H} \left[ \int_t^T \left\{ R_n \Phi(S_n) e^{H_n} \lambda_n \exp\left( - \int_t^n e^{H_s} \lambda_s ds \right) \right\} du / \mathcal{F}_t \right]
\]

and \( \mathbb{Q}_H = \mathbb{Q}_o \) on \( \mathcal{F}_o \) then

\[
1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_H} [R_t \Phi(S_t) 1_{\{\tau < T\}} / \mathcal{G}_t] = 1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_o} \left[ \int_t^T \left\{ R_n \Phi(S_n) e^{H_n} \lambda_n \exp\left( - \int_t^n e^{H_s} \lambda_s ds \right) \right\} du / \mathcal{F}_t \right]
\]

For \( e^{H_n} = -1 + \frac{1}{n} \),

\[
\lim_{n \to +\infty} 1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_o} \left[ \int_t^T \left\{ R_n \Phi(S_n) e^{H_n} \lambda_n \exp\left( - \int_t^n e^{H_s} \lambda_s ds \right) \right\} du / \mathcal{F}_t \right] = 0.
\]

and therefore \( 0 = \inf_{\mathbb{H}} 1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_H} [R_t \Phi(S_t) / \mathcal{G}_t] \).

For the upper bound, we first remark that

\[
1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_o} \left[ \int_t^T \left\{ R_n \Phi(S_n) e^{H_n} \lambda_n \exp\left( - \int_t^n e^{H_s} \lambda_s ds \right) \right\} du / \mathcal{F}_t \right] \leq 1_{\{t \leq \tau\}} R_t \sup_x \Phi(x) \mathbb{E}^{\mathbb{Q}_o} \left[ \int_t^T e^{H_n} \lambda_n \exp\left( - \int_t^n e^{H_s} \lambda_s ds \right) \right] du / \mathcal{F}_t \]

\[
\leq 1_{\{t \leq \tau\}} R_t \sup_x \Phi(x)
\]

From expression 17, we obtain

\[
\sup_{\mathbb{H}} 1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_H} [R_t \Phi(S_t) 1_{\{\tau < T\}} / \mathcal{G}_t] \leq 1_{\{t \leq \tau\}} R_t \sup_x \Phi(x)
\]

Moreover, since assumption 5 is valid under \( \mathbb{Q}_H \) one has

\[
1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_H} [R_t \Phi(S_t) / \mathcal{G}_t] - R_t \sup_x \Phi(x)
\]

\[
= 1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_o} \left[ \int_t^T \left\{ R_n \Phi(S_n) - R_t \sup_x \Phi(x) (1 - \exp\left( - \int_t^n e^{H_s} \lambda_s ds \right) \right\} e^{H_n} \lambda_n \exp\left( - \int_t^n e^{H_s} \lambda_s ds \right) \right] du / \mathcal{F}_t
\]

For \( e^{H_n} = n \), then for all \( t \),

\[
\lim_{n \to +\infty} \mathbb{E}^{\mathbb{Q}_o} \left[ \int_t^T \left\{ R_n \Phi(S_n) - R_t \sup_x \Phi(x) (1 - \exp\left( - \int_t^n e^{H_s} \lambda_s ds \right) \right\} e^{H_n} \lambda_n \exp\left( - \int_t^n e^{H_s} \lambda_s ds \right) \right] du / \mathcal{F}_t
\]

0.

and we conclude that \( \sup_{\mathbb{H}} 1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_H} [R_t \Phi(S_t) / \mathcal{G}_t] = 1_{\{t \leq \tau\}} R_t \sup_x \Phi(x) \).

ii) The proof is exactly the same as the one for the lower bound. For the upper bound, just note that \( 0 \leq \Phi(x) \leq x \) implies

\[
1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_H} [R_t \Phi(S_t) 1_{\{\tau < T\}} / \mathcal{G}_t] \leq 1_{\{t \leq \tau\}} \mathbb{E}^{\mathbb{Q}_H} [R_t S_t 1_{\{\tau < T\}} / \mathcal{G}_t] = 1_{\{t \leq \tau\}} R_t S_t
\]

\( \text{(RS is a martingale under } \mathbb{Q}_H \text{).} \)
Then
\[
\sup_{t \leq \tau} 1_{\{t \leq \tau\}} \mathbb{E}^{\mathcal{Q}_t} \left[ R_t \Phi(S_t) 1_{\{t < \tau\}} / \mathcal{G}_t \right] \leq 1_{\{t \leq \tau\}} R_t S_t.
\]
and we conclude as previously.

A.5 Proof of Proposition 6
The probability \( \mathcal{Q}_o \) satisfies i) and ii). So it is enough to show point iii).

Let \( Y \) a \( \mathcal{P} - \mathcal{G} \)-square martingale, orthogonal to the \( W \) under \( \mathcal{P} \). From a representation theorem from Kusuoka (1999), one has that
\[
dY_t = Y_0 + \int_0^t \xi_s dM_s, \quad \text{with} \quad \mathbb{E}^\mathcal{P} \left[ \int_0^\infty |\xi_t|^2 \lambda_t dt \right] < +\infty.
\]
As \( M \) is a \( \mathcal{G} \)-martingale under \( \mathcal{Q}_o \), \( Y \) is also a \( \mathcal{G} \)-martingale under \( \mathcal{Q}_o \).

A.6 Proof of Proposition 7
Let \( \mathcal{Q}_H \) be an EMM. One has that
\[
I(\mathcal{Q}_H, \mathcal{P}) = \mathbb{E}^{\mathcal{Q}_H} \left( \frac{d\mathcal{Q}_H}{d\mathcal{P}} \ln \frac{d\mathcal{Q}_H}{d\mathcal{P}} \right) = \mathbb{E}^{\mathcal{Q}_H} (\ln \frac{d\mathcal{Q}_H}{d\mathcal{P}})
\]
\[
= \mathbb{E}^{\mathcal{Q}_H} \left[ \int_0^\infty \left( \frac{d\mathcal{P}}{d\mathcal{Q}_H} \right) dt + (1 - N_{t-}) (e^{H_t} H_t - (e^{H_{t-}} - 1)) \lambda_t dt \right]
\]
\[
= \mathbb{E}_\mathcal{P} \left[ \int_0^\infty \frac{d\mathcal{P}}{d\mathcal{Q}_H} dt + \mathbb{E}^{\mathcal{Q}_H} \left[ \int_0^\infty (1 - N_{t-}) (e^{H_t} H_t - (e^{H_{t-}} - 1)) \lambda_t dt \right] \right]
\]
To conclude, we just remark that the function \( H \rightarrow e^{H_t} H_t - (e^{H_{t-}} - 1) \) is positive, and therefore
\[
\mathbb{E}^{\mathcal{Q}_H} \left[ \int_0^\infty (1 - N_{t-}) (e^{H_t} H_t - (e^{H_{t-}} - 1)) \lambda_t dt \right] \geq 0 = \mathbb{E}^{\mathcal{Q}_o} \left[ \int_0^\infty (1 - N_{t-}) (e^0 \times 0 - (e^0 - 1)) \lambda_t dt \right]
\]

A.7 Proof of Theorem 1
We provide a proof with \( r = 0 \) for simplicity. We have
\[
M_t^{\lambda^X} = 1_{\{t \geq \tau\}} X_t + \mathbb{E}^{\mathcal{Q}_H} \left[ \frac{X_t 1_{\{t < \tau\}}}{\mathbb{E}^{\mathcal{Q}_H} [1_{\{t < \tau\}} | \mathcal{F}_t]} \right] 1_{\{t < \tau\}}
\]
Then, as assumption 3 holds under \( \mathcal{Q}_H \), \( Z_t := \mathcal{Q}_H(t < \tau | \mathcal{F}_t) = e^{\lambda^X} = \mathcal{Q}_H(t < \tau | \mathcal{F}_\infty) \), so that we have
\[
\mathbb{E}^{\mathcal{Q}_H} \left[ X_t 1_{\{t < \tau\}} | \mathcal{F}_\infty \right] = \int_t^\infty X_s e^{-\lambda^X_s} \lambda_s ds
\]
Therefore, remark 1 implies that
\[
\mathbb{E}^{\mathcal{Q}_H} \left[ \int_t^\infty X_s e^{-\lambda^X_s} \lambda_s ds | \mathcal{F}_t \right] = \mathbb{E}^{\mathcal{Q}_0} \left[ \int_t^\infty X_s e^{-\lambda^X_s} \lambda_s ds | \mathcal{F}_t \right]
\]
So
\[
M_t^{\lambda^X} = 1_{\{t \geq \tau\}} X_t + 1_{\{t < \tau\}} e^{\lambda^X} \mathbb{E}^{\mathcal{Q}_0} \left[ \int_t^\infty X_s e^{-\lambda^X_s} \lambda_s ds | \mathcal{F}_t \right]
\]
If we denote by \( \mu^{\lambda^X}_t = e^{\lambda^X} \mathbb{E}^{\mathcal{Q}_0} [ \int_t^\infty X_s e^{-\lambda^X_s} \lambda_s ds | \mathcal{F}_t ] \), one has
\[
\mu^{\lambda^X}_t = e^{\lambda^X} \mathbb{E}^{\mathcal{Q}_0} [ \int_t^\infty X_s e^{-\lambda^X_s} \lambda_s ds | \mathcal{F}_t ] - e^{\lambda^X} \int_0^t X_s e^{-\lambda^X_s} \lambda_s ds = e^{\lambda^X} J_t - e^{\lambda^X} \int_0^t X_s e^{-\lambda^X} \lambda_s ds.
\]
Itô’s formula implies

\[
d\mu_t^X = e^{\tilde{\lambda}_t}dJ_t + J_{t-}d(e^{\tilde{\lambda}_t}) - X_t\tilde{\lambda}_tdt - \left(\int_0^t X_s e^{-\tilde{\lambda}_s}\tilde{\lambda}_s ds\right)d(e^{\tilde{\lambda}_t})
\]

\[
= e^{\tilde{\lambda}_t}dJ_t + (\mu_{t-}^X - X_t)\tilde{\lambda}_tdt.
\]

Since \(\mu_t^X 1_{\{t<\tau\}} = J_0 + \int_0^{t\land \tau} d\mu_s^X - \mu_{t-}^X 1_{\{t>\tau\}}\) and \(\Delta \mu_t^X = \Delta J_t = 0\) (all \(\mathbb{F}\)-martingales are continuous), we conclude that

\[
M_t^X = J_0 + \int_0^{t\land \tau} e^{\tilde{\lambda}_s}dJ_s + \int_0^{t\land \tau} (\mu_s - \mu_{t-})\tilde{\lambda}_s ds + (X_{t\land \tau} - \mu_{t-}^X) 1_{\{t\geq \tau\}}.
\]

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