Grafting Information in Scenario Trees
Application to Option Prices

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1. Introduction

Many mathematical models used in management science do not impose any complex restrictions on the parameter values arising in a given problem. So, for example, the coefficients appearing in the objective function and in the constraints of a generic linear programming problem are essentially unrestricted; the distances defining an instance of the travelling salesman problem are only required to be nonnegative; etc. As a consequence, when the instances arise from a specific real-world application, the models under consideration are usually sufficiently robust to remain meaningful even if their numerical parameters are not estimated with very high accuracy or, alternatively, if small perturbations are applied to the parameter values.

Things can be quite different, however, with models developed for financial applications. Indeed, in such applications, many models turn out to be utterly meaningless unless one can ensure that the data sets are realistic and internally consistent with the assumptions underlying the model. For instance, a classical simplifying hypothesis in financial models consists in assuming that stock and index returns follow a normal probability distribution. However, we observed in several numerical experiments that this normality hypothesis, when used in conjunction with the option prices observed on the market or with carelessly generated option prices, leads almost inevitably to arbitrage opportunities and/or to abnormal (even infinite) expected returns in portfolio selection models. Moreover, the optimal portfolio returns are also very sensitive to slight perturbations in the value of some key parameters like the risk-free rate or the index dividend yield.

In view of these observations, the objective of this paper is to propose a coherent methodology which allows for the avoidance some of the pitfalls that are encountered when developing financial models. More precisely, the paper presents a methodology to realistically, consistently and flexibly model the distribution of future values of a security. The methodology relies on multi-period scenario-trees where every node represents a possible future state of the world. To instantiate the tree, historical or instantaneous implied probability density functions are first generated from security prices or current options prices, and are subsequently sampled.

In this article, we will present a specific application using a set of put and call options written on the S&P 500 index in order to illustrate the robustness of the approach. Information about the distribution of the underlying index is retrieved from a subset of option prices, processed and synthesised through the multinomial scenario-tree, and used again for the pricing of a set of option contracts. Through the use of the same set of options as inputs and outputs, we can verify the ability of the model to preserve initial information. On the other hand, the application of the results to a broader or to a distinct set of option contracts allows us to test the pricing performance of this approach. Our results indicate that these in-sample and out-of-sample criteria are both satisfied to an impressive extent, relatively to the use of historical distributions, when all information embedded in option prices is used to obtain an “implied distribution” of the index. The approach also compares very favourably with Rubinstein’s methodology (1994) (slightly adapted in order to ensure its comparability with our approach).

The paper is organised as follows. Section 2 briefly sums up the main advantages and difficulties associated with the use of scenario-trees in financial models. In Section 3, we recall how the put-call parity relation can be exploited in order to derive the implied value of the risk-free rate and of the dividend yield from the observed option prices, rather than from historical data. Section 4 applies a similar philosophy to the estimation of the density function of returns: here, we compare the results obtained by fitting classical parametric distributions to past data on the one hand, and by using the implicit information contained in option prices on the other hand. Section 5 describes the construction of scenario trees based on the previous approaches. Finally, in Section 6, we validate this methodology with some illustrative applications in the context of option pricing.
2. A Multinomial Tree Approach

Scenario-trees represent a natural method to model the future. Each node of a scenario-tree models a possible state of the world at a particular date. More precisely, a node is associated with the value of each security considered in the corresponding state of the world, and with the probability that this state occurs. A simple two-period tree is illustrated in Figure 1.

Figure 1: Two-period scenario-tree

Such trees provide very generic, though relatively simple, models to represent future states of the world in stochastic optimisation problems (see e.g. Birge and Louveaux (1999) or Prekopa (1995) for a broad introduction to stochastic programming).

Particularly in finance, trees of scenarios have been used in numerous computational models, both in applied and in theoretical frameworks, as in Dembo (1991), Dybvig (1988a, 1988b), Mulvey (1994) and Rubinstein (1994). As a matter of fact, binomial methods, which are intensively used in the derivatives pricing area, rely on special types of scenario trees.

Grinold (1999) ponders some of the relative advantages and drawbacks of scenario-based approaches vs. mean-variance approaches when dealing with portfolio optimisation problems. He writes: "Mean-variance has dominated in the investment management profession, while expected utility (scenario-based optimisation) is highly regarded in the academy. Today there is revived interest among professionals in moving beyond mean and variance to capture skewness, downside risk, and other high moments of portfolio returns."

In Grinold’s view, scenario-based maximisation is mostly useful — and even indispensable — for portfolio management problems involving options or assets with alternative distributions. It deals with the entire distribution of outcomes and thereby allows for a broad variety of objectives. However, it also has to respond to several major challenges. In particular, the approach requires the specification of the entire distribution of all assets, and still to work within a realistic framework.

Instead, we view the exhaustive characterisation of the dynamics of all assets as an effective way to encode highly valuable information, rather than a source of useless and unmanageable complexity. A consistent use of these distributions, which compress relevant information without any significant loss in the process, could eventually yield results of unprecedented parsimony and quality.

To handle this challenge, we consider an extended tree approach where all sub-trees could be interpenetrated or not, with or without recombining leaves, with equiprobable or non-equiprobable nodes and with a large or a small number of leaves. Such multinomial trees of scenarios constitute a larger family than the classical family of binomial trees.
Practically, we handle the inherent complexity of this broad, general framework with an approach using moment-based and implied characterisations of probability density functions together with stratified sampling. The first technique enables us to obtain a remarkable trade-off between parsimony and accuracy, while the second one preserves information relative to the distributions in the parameterisation of the tree. Thanks to this modelling choice, only a relatively small number of nodes is required to faithfully represent lots of complex continuous problems.

3. Implied Basic Parameters

3.1 Methodology
Before density functions can be handled, we need to obtain realistic and consistent values of some basic parameters characterising securities on a market: in particular, we need estimates of the risk-free rate ($r$) and of the continuous dividend yield ($q$) of each security. Most of these parameters are directly available or can be recomputed from financial databases, but the estimates can be sensitive to various factors, e.g. to the specific market or to the periodicity of data collection. This may introduce undesirable noise or leads and lags in the parameters. To avoid these pitfalls, we find it preferable to rely on implied values derived from currently observed options prices. The approach is similar to the use of implied volatilities obtained by inverting the Black-Scholes formula when the returns of the underlying are assumed normally distributed (see Hull (1997) for several possible schemes), but will be extended to a larger set of parameters and a broader option pricing framework.

In order to estimate the risk-free rate ($r$) and the dividend yield ($q$), Shimko (1993) proposes to exploit the put-call parity equation:

$$c - p = S e^{-q(T-t)} - X e^{-r(T-t)},$$

where $c$ is the price of a call and $p$ is the price of a put with the same time-to-maturity ($T-t$) and the same strike price ($X$), and $S$ is the current price of the underlying security. If the parity equation held exactly, then only two pairs of put-call options would be required in order to derive the parameters of interest ($r$ and $q$) from (1). In practice, however, and especially for options with a short time to maturity and some options far from the money, deviations from equality may appear in (1). Therefore, a more robust strategy is to estimate $r$ and $q$ by performing a linear regression on the price differences $c - p$ with respect to the corresponding strike prices $X$, for a set of options around the money and with the same time-to-maturity $T-t$.

Note that this procedure only uses instantaneous available information (options with time-to-maturity in the future) and does not require any historical data. It can be effectively used when time-to-maturity is not too short; otherwise numerical instabilities could appear.

3.2 Application
As an illustrative example throughout this paper, we will consider the case of an investor who plans to invest in S&P500 options on 23 September, 2002. His investment horizon is one month, and he only considers options with the corresponding maturity. In order to construct a model representing the evolution of the S&P500 index over this period, he first needs to obtain the risk free rate $r$ and the dividend yield $q$.

The corresponding annualized T-bill rate at this time is 1.68% and, according to the CBOE database, the dividend yield is 1.69%. The investor could trust these values but since he intends to trade S&P500 options, another available strategy is to observe the current price of these options and to use equation (1) to derive the implied and instantaneous rates.
In order to perform a regression (see Figure 2) on (1), a sample of options is selected so as to minimise the difference with the most recent official T-bill rate (Issue date: 09-19-2002) with a term of 28 days. This leads to a risk-free rate of 1.78% and a dividend yield of 2.16% obtained from 25 pairs of options.

These results clearly indicate that the nature of the information that is derived from this option-based, “forward-looking” approach is inherently different from the historical information contained in past observations. In Section 6, we shall try to assess whether or not this information is reliable.

For the time being, and in view of the small difference between the values obtained for the risk-free in both approaches, we consider that the set of 25 selected options is representative of the market and we use it subsequent examples.

Figure 2: SP500: parity equations

4. Probability Density Functions

4.1 Moment-based approaches

4.1.1 Normal distribution

The normal distribution is frequently used to obtain a basic, benchmark representation of the distribution of returns of a security. The reliance on this distribution is essentially justified by its simplicity – it is completely characterised by its first two moments, i.e. mean and standard deviation – and on theoretical grounds. Also, whenever other tools based on the normality assumption are simultaneously used (e.g. the Black-Scholes formula), then the normal distribution has the merit to preserve the internal consistency of the models. However, in spite of these advantages, the normal distribution is well-known to provide a rather poor fit of real-world observed returns, and alternative models have therefore been proposed in the literature.

4.1.2 Four-moment models

Depending on the market and on the data collection period, returns can exhibit large deviations from the normal distribution. In particular, a kurtosis and a skewness effect are frequently reported in empirical studies. In order to account for these effects, various extensions of Student’s t pdf have been introduced.

For instance, Theodossiou (1998) suggests to split the density function in two areas around the null mode, and to define the density

\[
 f(x|k, n, \lambda, \sigma^2) = \begin{cases} 
 f_1 = C(1 + \frac{k}{n-2}(\frac{|x|}{(1-\lambda)\theta\sigma}))^k \frac{(n+k)}{k} & \text{for } x < 0 \\
 f_2 = C(1 + \frac{k}{n-2}(\frac{|x|}{(1+\lambda)\theta\sigma}))^k \frac{(n+k)}{k} & \text{for } x \geq 0 
\end{cases}
\]
where $C$ and $\theta$ are normalizing constants ensuring that $f(\cdot)$ is a proper density function (see Theodossiou (1998) for details), $k$ and $n$ determine the height and tails, $\lambda$ controls skewness ($f_1$ and $f_2$ have identical expressions except for the coefficient $(1 \pm \lambda)$) and $\sigma^2$ is the variance.

In another paper, Fernandez and Steel (1998) have proposed a general method to produce a skewed variant of an arbitrary unimodal and symmetric density functions. The basic idea is simply to introduce a scaling factor in the negative orthant and its inverse in the positive orthant. They illustrate the approach for Student’s t distribution, to which they associate the skewed pdf $f(x|m, s^2, \nu, \gamma)$

$$f(x|m, s^2, \nu, \gamma) = \begin{cases} \frac{2}{\gamma + \frac{1}{2}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi} \nu^2 \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{(x-m)^2}{s^2 \nu^2}\right)^{-\frac{(\nu+1)}{2}} & \text{for } x - m \geq 0 \\ \frac{2}{\gamma + \frac{1}{2}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi} \nu^2 \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{(x-m)^2}{s^2 \nu^2}\right)^{-\frac{(\nu+1)}{2}} & \text{for } x - m < 0 \end{cases}$$

where $m$, as the mode, only models the location, $s^2$ only models the dispersion, $\gamma$ (> 0) only models skewness and $\nu$ (> 0) only models kurtosis. The value of the coefficients ensures that $f(\cdot)$ is a proper pdf. When $\gamma$ is not equal to one, the mode is preserved, but the skewness is modified. In this approach, each moment of the distribution is fully specified by a specific parameter. It is therefore easier to interpret and to handle than Theodossiou’s density function.

Yet, both approaches share a common drawback: the parameters of the distributions are difficult to estimate numerically. In order to apply maximum likelihood techniques, for instance, large data sets are required. But long time series, digging far into the past, seem to offer little guarantee for a reliable representation of the distribution of future returns.

Moreover, in some cases, even four moments may prove insufficient to provide an adequate model of the market distribution of returns. As a matter of fact, Rubinstein (1994) and Shimko (1993) display examples of observed market distributions featuring two modes, that cannot be represented by the above parametric models. The next section presents an alternative approach which allows to address these problems.

4.2 Implied models

Observed option prices contain a wealth of relevant information about market prices, as they reflect the investors’ expectations about future moves. Moreover, they can be directly observed at the precise time when an investment decision is to be made. Therefore, just as in Section 3, we would like to extract as much implied information as possible from these prices.

In fact, based on current option prices, it is possible to construct a complete pdf of future returns without assuming any prior (parametric) shape for the pdf. Furthermore, the approach to be described ensures internal coherence between the observed option prices and the return distribution of the underlying in the model.

The relationship between option prices and the pdf ($f$) or the cumulative distribution function ($F$) of the underlying security returns has been elicited by Breeden and Litzenberger (1978). In order to state their result, we first need to express the option price as a continuous function $C(X)$ of the strike price $X$: there holds (Cox and Ross (1976))

$$C(X) = e^{-r(T-t)} \int_X^\infty (S - X) f(S) dS,$$  

(4)
and hence

\[
\frac{\partial C(X)}{\partial X} = -e^{-r(T-t)} (1 - F(X)), \quad (5)
\]

\[
\frac{\partial^2 C(X)}{\partial X^2} = e^{-r(T-t)} f(X). \quad (6)
\]

The equality (6) implies that the pdf \( f(X) \) can be deduced from the pricing function \( C(X) \).

In order to obtain an analytical expression of \( C(X) \), we could for instance interpolate the option (call) prices \( C(X_i) \) observed on the market for a discrete sample of strike prices \( X_i \) \((i = 1, 2, \ldots, m)\). Shimko (1993) proposes another approach that allows to take the smile effect into account. He first computes the implied volatilities \( \sigma(X) \) of the observed calls by inverting the Black-Scholes formula. Then he fits the observations with a quadratic function (by the least-square method) so as to obtain an analytical expression of \( \sigma \) as a function of the strike price \( X \):

\[
\sigma(X) = A_0 + A_1 X + A_2 X^2. \quad (7)
\]

Applying the Black-Scholes pricing formula with this expression of \( \sigma(X) \) (instead of a constant \( \sigma \)) yields the continuous option pricing function

\[
C(X) = S e^{-\sigma(T-t)} N(d_1) - X e^{-r(T-t)} N(d_2) \quad (8)
\]

where

\[
d_1 = \frac{\ln \left( \frac{S}{X} + (r - q + \frac{\sigma^2(X)}{2}) \right)}{\sigma(X) \sqrt{T-t}}, \quad (9)
\]

\[
d_2 = d_1 - \sigma(X) \sqrt{T-t}, \quad (10)
\]

and \( N(\cdot) \) is the cumulative normal distribution. The pdf \( f(X) \) can now be analytically derived from (6) and (8), even though the resulting expression is rather complex.

Note that in this approach, the BS formula is only used as a conversion tool from option prices to volatilities (and conversely). Therefore, the approach is quite robust under small deviations from the assumptions underlying the BS formula (see Shimko (1993)).

On the other hand, it is important to note that the above procedure rests on two strong assumptions concerning respectively the nature of the pdf to be computed and the domain of validity of (8).

The first pdfs presented in Section 4.1 (normal, skewed Student, etc.) implicitly referred to consensus subjective probabilities: they model the “real” market, as perceived by the investors. On the contrary, the pdf in (6) is defined in a risk-neutral world. Therefore, further steps are required to convert this risk-neutral pdf into a consensus pdf, so as to be able to instantiate the tree of scenarios.

Another difficulty is that the pdf (8) can only be used in the range of the observed strike prices, since the regressed relation (7) cannot really be trusted outside this range (see Shimko (1993)). In view of this, the largest possible range of option strike prices should be used when estimating the volatility (7). But still, the problem usually remains to define the tails of the distribution so as to preserve the properties of a proper risk-neutral pdf (i.e. sum equal to one, smooth function without breakpoints, expected value equal to the risk-free rate).
This poses complex numerical problems which are not completely overcome in Shimko (1993). In our implementation, we assume that each tail of the implied pdf corresponds to the tail of a normal distribution (a different distribution for each tail), for which we numerically determine appropriate values of the parameters.

4.3 Application
We return to the small case-study introduced in Section 3.2, with the objective to construct the pdf of the S&P500 monthly returns.

In order to construct either a normal pdf or Fernandez-Steel's skewed version of the Student pdf (see Figure 3), we need to estimate the first (two or four) moments of the distribution of future returns. These moments can be simply computed from a sample of past returns by classical statistical techniques.

Note, however, that the distribution of the S&P500 monthly returns does not appear to be stationary (see Figure 3a for a plot of the observed returns over a 10-year period). In Figures 3b to 3d, we display the evolution of the first four moments over the same period. More exactly, data points in Figure 3b represent the estimated value of the mean returns and of the standard deviation computed over the previous year. Similarly, Figures 3c and 3d represent the estimated value of the skewness and kurtosis, each of them smoothed over the preceding year. We could rely on a smoothing function to obtain an accurate estimate of each forecasted moment.

Yet, very old observations of a long time series are unlikely to provide reliable forecasts of the immediate future, especially for a highly nonstationary time series like the S&P500 one. For the purposes of our application, the use of a simple smoothing function would thus yield overly inaccurate estimates.

*Figure 3: Returns and moments*
In order to overcome this non-stationarity issue, we have developed a simple algorithm to capture the moments more faithfully. Firstly, we construct a time series of monthly values for each moment. Each observation is obtained by using the lagged monthly returns of the previous year, i.e. with twelve lags. Secondly, we adjust, for each smoothed moment, a regression by a polynomial of degree four.

This degree ensures that residuals are of a small magnitude. Based on these regressions, we can compute a forecast of the four moments one month later. Thirdly, we compare the actual monthly value with the polynomial forecast. Our best estimate of the predicted value is then obtained by averaging the analytical forecast and the actual value. For each moment, the optimal weight \( w \) minimises the squared difference between the observed value \( x_t \) at time \( t \) and the weighted average of the lagged value \( x_{t-1} \) and the adjusted interpolated value \( f_t \) over the sample period:

\[
\min \sum (x_t - (wx_{t-1} + (1-w)f_t))^2
\]

The optimal weight \( w^* \) is given by:

\[
w^* = \frac{\sum (x_t - 1)(x_{t-1} - f_t)}{\sum (x_{t-1} - f_t)^2}
\]

The estimated value of the four moments at time \( t+1 \) is reported in Column 1 of Table 1. Based on these values, the normal and skewed Student pdf can be defined. More exactly, the normal pdf is determined by the values of the mean and the standard deviation only, while its skewness and kurtosis are predetermined. For the skewed Student pdf, we fix again the mean and standard deviation at their empirical values, and the parameters \( \gamma \) and \( \upsilon \) in Eq. (3) are optimised so that the third and fourth moments of the distribution match as closely as possible their empirical values. In this way, an excellent fit is obtained for the skewness coefficient (0.06129 vs. 0.05976), but not for kurtosis. Actually, it can be shown that the skewed Student distribution is leptokurtic.

The results are displayed in Table 1 and Figure 4. In this case, there is almost no noticeable difference between the normal distribution and the skewed Student distribution.

Let us now turn to the estimation of the implied pdf via equations (6)-(8). We first need to select an appropriate set of call options which will be used to compute the volatility function \( \sigma(X) \). Since equations (6)-(8) also involve estimates of the risk-free rate and of the dividend yield, it is coherent to use the same set of 25 options as in Section 3.2. We thus obtain an expression of the
implied pdf between the two extreme strike prices. This expression represents 88% of the distribution. In order to complete the description of the distribution, we fit two normal distributions on the tails. The first moments of the resulting implied pdf are given in Table 1 under the heading “Implied (rn)” (reminding us that it is defined in a risk-neutral world) and the pdf is drawn in Figure 4.

In order to be able to compare the moments of the pdfs in Table 1, we have transformed the risk-neutral pdf into a real one by a utility-based approach (this is typically done through the log-utility function). This yields the last column of Table 1. Here again, as in Section 3.2, we observe that the implied pdf is quite different from the pdf derived from past observations.

Table 1: Moments of the distributions

<table>
<thead>
<tr>
<th></th>
<th>Empirical</th>
<th>Normal</th>
<th>Skewed t</th>
<th>Implied (rn)</th>
<th>Implied (real)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>-0.01887</td>
<td>-0.01887</td>
<td>-0.01887</td>
<td>-0.00414</td>
<td>0.00612</td>
</tr>
<tr>
<td>Standard dev.</td>
<td>0.05090</td>
<td>0.05090</td>
<td>0.05090</td>
<td>0.10441</td>
<td>0.09851</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.05976</td>
<td>0</td>
<td>0.06129</td>
<td>-1.18245</td>
<td>-1.14846</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>1.78132</td>
<td>3</td>
<td>2.97420</td>
<td>5.39984</td>
<td>5.43135</td>
</tr>
</tbody>
</table>

Quite remarkably, the empirical and the implied distributions exhibit two main differences playing in opposite directions: the mean return of the empirical distribution is much lower than the mean of the implied distribution, but all risk parameters are more favorable (lower variance and kurtosis, higher skewness). This probably indicates that investors, through their trades on the option market, are more optimistic regarding the future rate of return of the index than what they currently observe; conversely, they relate this higher expected return — although still negative — to higher risk exposure.

5. Sampling

Constructing a tree of scenarios requires an appropriate discretisation method which allows to represent a continuous distribution by a small sample of its values. For this purpose, we use the “stratified sampling” approach (see e.g. Hull (1997); other authors use the name “stylised sampling”; see Hampel et al. (1986)).

In order to stratify the continuous distribution, the area under its density curve is partitioned into the desired number of equiprobable intervals (say, nbS intervals), where each interval corresponds to one leaf of the tree of scenarios. Thus, all the leaves are equiprobable and each future state of the world has the same weight.

In each interval, one value is selected to represent the whole interval. Various choices are available here: the mean of the interval, its median, the mean of its bounds, etc. Note that each possibility has its own advantages and drawbacks. For example, in the case of a symmetric distribution, selecting the mean of each interval preserves the global mean. However, computing the mean may be computationally demanding.

The stratification process is illustrated in Figure 5. The advantage of this method is that, when the size of the sample increases, the moments of the sample converge much faster to the corresponding continuous values than with a Monte Carlo approach. It is therefore possible to faithfully represent any distribution with relatively small sets of data and small trees of scenarios. For the data set we consider, this approach already leads to very good results with samples of nbS = 50 scenarios (see Table 2 and compare with Table 1).
6. Option pricing

In previous sections, we have explained how a sample of the pdf of future security prices can be derived from the implicit information contained in option prices. We now present two simple validations of this approach. Namely, our objective will be to demonstrate that the resulting tree of scenarios allows to price options in a realistic and consistent fashion, i.e. it allows to reconstruct faithful estimates of the observed market prices and it compares favorably with an alternative approach proposed by Rubinstein (1994).

6.1 Comparisons of the distributions

Suppose that we use a tree of scenarios to model a decision problem unfolding over the horizon $t, t+1, \ldots, T$ (see Figure 1). The model involves a set of options defined over a single underlying security, with maturity at time $T$. Each leaf of the tree can be instantiated with one of the possible prices of the security at time $T$, as explained in previous sections. When this is done, the value of each option at maturity is readily available for each of the nbS scenarios at hand. Hence, the tree can in principle be used to price the options at time $t$, by discounting the final option values in a risk-neutral world. The approach is well-known when the tree of scenarios is binomial, but is more complex with multinomial trees. To understand this, observe that the option prices at time $t$ are classically derived from the arbitrage equations:

$$
\begin{pmatrix}
1 \\
S_0 \\
p_{opt}^1 \\
\vdots \\
p_{opt}^{nbO}
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\vdots \\
\psi_{nbS}
\end{pmatrix}
= 
\begin{pmatrix}
S_0 e^{r(T-t)} & \ldots & S_{nbO} e^{r(T-t)} \\
S_1 e^{q(T-t)} & \ldots & S_{nbO} e^{q(T-t)} \\
p_{opt}^{1,1} & \ldots & p_{opt}^{1, nbS} \\
\vdots & \vdots & \vdots \\
p_{opt}^{nbO,1} & \ldots & p_{opt}^{nbO, nbS}
\end{pmatrix}
\begin{pmatrix}
e^{r(T-t)} \\
e^{r(T-t)} \\
e^{r(T-t)} \\
\vdots \\
e^{r(T-t)}
\end{pmatrix}$$

where $nbO$ is the number of options in the model, nbS is the number of scenarios in the tree, $r$ is the risk-free rate, $q$ is the dividend yield, $S_0$ is the initial price of the underlying asset, $S_j$ is the final price of the asset in scenario $j$, $p_{opt}^i$ is the initial price of option $k$, $p_{opt}^{i,j}$ is the final value
of option \( k \) in scenario \( j \), and \( \tilde{\psi}_j \geq 0 \) is the Arrow-Debreu state-price attached to scenario \( j \).

The notation \( \tilde{x} \) indicates that \( x \) is an unknown. Since there are usually more unknowns than equations in (11) (i.e. \( n_{bS} >> 2 \)), the option prices are not completely determined in the solution of the system. We can use this degree of freedom to select a solution of the arbitrage equations which reproduces as closely as possible the observed market prices at time \( t \). This objective can be expressed by an optimisation model which minimises the sum of squared relative errors between the computed option prices and appropriate target values:

\[
\begin{align*}
\text{minimize} & \quad \sum_{k=1}^{n_{bO}} \left( \frac{\text{popt}_k - \text{target}_k}{\text{target}_k} \right)^2 \\
\text{subject to} & \quad \text{arbitrage equations (11)} \\
& \quad \text{popt}_k \geq 0 \quad (n_{bO} = 1, \ldots, k) \\
& \quad \tilde{\psi}_j \geq 0 \quad (n_{bS} = 1, \ldots, j).
\end{align*}
\]

Relative (or weighted) differences are used in this model to avoid that some options — especially those deeply out-of-the-money and less liquid — influence disproportionately the results.

Let us return again to the small numerical application that we have already handled in Sections 3.2 and 4.3. On the initial date (23 September 2002), 33 calls and 46 puts with a one-month maturity were available. So far, we have used 25 calls to estimate the parameters \( r \) and \( q \) and to define the implied pdf. Now, we would like to check whether the various scenario-trees built in Section 5 can be used to reproduce the target option prices, including those of the options which have not been used in the construction of the implied pdf. For each option, the target price is set equal to the mean of the market bid and ask prices.

Therefore the pricing model (12) has been optimised for different sets of options and different pdfs. The results are summarised in Table 3, which displays the mean, median and standard deviation of the relative differences between optimal and observed prices.

Note that, for practical purposes, it is usually not necessary to reproduce exactly the target price. Indeed, if the price arising from model (12) falls between the observed bid price and ask price, then this optimal price is coherent with the real market: indeed, we can add some spread around it so as to match the observed prices without losing the no-arbitrage property. For each case, the last lines of Table 3 give the proportion of option prices falling between the bid and ask prices.

As it clearly appears from Table 3, the implied distribution leads to better results than the other distributions. In particular, the results obtained with the implied pdf are nearly perfect for the initial set of 25 calls. The loss of performance for the other data sets appears to be due mostly to the eight calls far out-of-the-money.

6.2 Comparison with Rubinstein’s method
In his famous paper on implied trees, Rubinstein (1994) proposes another method to construct an implied pdf. Therefore it is interesting to compare the results of our approach with those obtained from Rubinstein’s.

Let \( S_b \) (\( S_a \)) be the current bid (ask) price of the underlying asset, and \( C_k^{b} \) (\( C_k^{a} \)) the current bid (ask) price for a call \( k \) maturing at time \( T \). If we know a prior guess \( P_j^0 \) of the risk-neutral probability for each scenario \( j \), then Rubinstein suggests to obtain a discrete representation \( \{ \tilde{P}_j = e^{r(T-t)} \tilde{\psi}_j \} \) of the risk-neutral pdf by optimising the following model:
In order to setup this model, we need to sample future asset values \( S_j \) and prior guesses for the risk-neutral probabilities \( (j = 1, \ldots, nbS) \). For this purpose, Rubinstein suggests to construct a classical multi-period binomial tree and to use the asset values and probabilities computed at the final leaves.

\[
\minimize \sum_{j=1}^{nbS} (e^{r(T-t)} \hat{\psi}_j - P^j_{\hat{\psi}})^2 \\
\text{subject to} \quad \begin{cases} 
S^b \leq \hat{S} \leq S^a \\
C^b_k \leq \text{pop} t_k \leq C^a_k \\
\text{pop} t_k \geq 0 \\
\hat{\psi}_j \geq 0 \end{cases} \quad (nbO = 1, \ldots, k) \\
(nbS = 1, \ldots, j).
\]

(13)

This approach rests on several assumptions and entails several limitations, as compared to ours. First, the computation of the initial guesses \( P^j_{\hat{\psi}} \) is based on a standard binomial tree which is implicitly linked to normality assumptions. The question arise whether this has an impact on the resulting \( \hat{P}_j \). According to Rubinstein, if a solution of (13) exists and other things being equal, then the denser the set of options, the less sensitive \( \hat{P}_j \) will be to the prior guess. However, when we deal with reasonably large number of scenarios, the number of unknowns exceed very much the number of constraints in the arbitrage equations (i.e. options) and the results become sensitive to the prior guesses.

Secondly, Rubinstein’s approach does not yield a complete continuous pdf, but rather a discrete set of risk-neutral probabilities. Actually, the asset values and associated probabilities describing the scenarios are uniquely defined by the underlying binomial tree. This provides less information than our approach, whereby we first produce the continuous implied distribution and subsequently

| Table 3: Relative differences between observed and optimised prices |
|---------------------------------|----------------|----------------|
|                                | Normal         | Skewed t       | Implied        |
| 25 calls                        |                |                |
| Mean                            | 27.63%         | 23.50%         | 1.46%          |
| Median                          | 5.30%          | 2.09%          | 0.14%          |
| Std. Dev.                       | 40.16%         | 39.78%         | 2.96%          |
| Between Bid-Ask                 | 8/25           | 12/25          | 25/25          |
| 25 calls and 25 puts            |                |                |
| Mean                            | 24.69%         | 23.22%         | 0.85%          |
| Median                          | 2.54%          | 2.61%          | 0.12%          |
| Std. Dev.                       | 40.22%         | 38.61%         | 2.17%          |
| Between Bid-Ask                 | 17/50          | 17/50          | 50/50          |
| 8 calls and 21 puts (strike prices not in the initial subset) |                |                |
| Mean                            | 27.69%         | 27.69%         | 15.63%         |
| Median                          | 0.04%          | 0.04%          | 0.03%          |
| Std. Dev.                       | 45.42%         | 45.42%         | 34.94%         |
| Between Bid-Ask                 | 19/29          | 19/29          | 24/29          |
| 33 calls                        |                |                |
| Mean                            | 41.63%         | 39.60%         | 16.17%         |
| Median                          | 8.69%          | 8.86%          | 0.25%          |
| Std. Dev.                       | 47.39%         | 46.26%         | 35.62%         |
| Between Bid-Ask                 | 8/33           | 7/33           | 29/33          |
| 33 calls and 46 puts            |                |                |
| Mean                            | 25.79%         | 24.86%         | 7.53%          |
| Median                          | 1.68%          | 1.86%          | 0.09%          |
| Std. Dev.                       | 41.94%         | 41.01%         | 24.29%         |
| Between Bid-Ask                 | 36/79          | 36/79          | 73/79          |
sample values and probabilities based on this knowledge. For example, it is not possible to obtain an equiprobable tree with Rubinstein's approach.

Finally, there is no guarantee that Rubinstein's model (13) is feasible. Indeed, his model assumes that there is no arbitrage opportunity for the scenarios representing the future market and the set of calls under consideration. But this assumption does not necessarily hold, and is actually violated for the set of 25 calls that we consider. In such a case, it is not possible to find state-prices such that all the option prices lie between the observed bid and ask prices.

In order to avoid infeasible solutions in (13) and to allow comparisons with our model, we have slightly adjusted Rubinstein's model. Namely, we have relaxed the constraint on the option spread \((C^p_k \leq \text{popt}_k \leq C^b_k)\), and we have modified the objective function so as to penalise any deviation from the observed spread.

Now, the experimental comparison of models can be carried out as follows. Based on the implied distribution on one hand, and the standard binomial tree on the other hand, we obtain two different scenario trees representing the future market. Then, using the same 25 calls in both cases, we solve model (12) with respect to the first set and Rubinstein's adjusted model with respect to the second set of scenarios, respectively, in order to obtain the state-prices associated to the leaves of each scenario-tree. Relying on the arbitrage equations, we can directly use these state-prices to price any other option observed on the market.

The results of these comparisons are displayed in Table 4. The two methods yield very close results even though they are based on quite different premises.

<table>
<thead>
<tr>
<th>Table 4: Relative differences between observed and computed prices</th>
<th>Implied pdf</th>
<th>Rubinstein's adjusted model</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>25 calls</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>1.46%</td>
<td>1.90%</td>
</tr>
<tr>
<td>Median</td>
<td>0.14%</td>
<td>0.00%</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>2.96%</td>
<td>4.36%</td>
</tr>
<tr>
<td>Between Bid-Ask</td>
<td>25/25</td>
<td>2/25</td>
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<tr>
<td><strong>25 calls and 25 puts</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.85%</td>
<td>1.06%</td>
</tr>
<tr>
<td>Median</td>
<td>0.12%</td>
<td>0.13%</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>2.17%</td>
<td>2.26%</td>
</tr>
<tr>
<td>Between Bid-Ask</td>
<td>49/50</td>
<td>49/50</td>
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<tr>
<td><strong>8 calls and 21 puts (strike prices not in the initial subset)</strong></td>
<td></td>
<td></td>
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<tr>
<td>Mean</td>
<td>19.24%</td>
<td>18.03%</td>
</tr>
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<td>Std. Dev.</td>
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<td>7.95%</td>
</tr>
<tr>
<td>Between Bid-Ask</td>
<td>22/29</td>
<td>24/29</td>
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<tr>
<td><strong>33 calls</strong></td>
<td></td>
<td></td>
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<tr>
<td>Mean</td>
<td>16.11%</td>
<td>16.52%</td>
</tr>
<tr>
<td>Median</td>
<td>0.25%</td>
<td>0.05%</td>
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<tr>
<td>Std. Dev.</td>
<td>35.66%</td>
<td>35.82%</td>
</tr>
<tr>
<td>Between Bid-Ask</td>
<td>29/33</td>
<td>28/33</td>
</tr>
<tr>
<td><strong>33 calls and 46 puts</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>7.74%</td>
<td>7.41%</td>
</tr>
<tr>
<td>Median</td>
<td>0.09%</td>
<td>0.08%</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>24.27%</td>
<td>24.29%</td>
</tr>
<tr>
<td>Between Bid-Ask</td>
<td>72/79</td>
<td>73/79</td>
</tr>
</tbody>
</table>
7. Conclusions
The approach of scenario trees in finance, and especially in derivatives pricing, is usually considered as promising but also quite challenging. The ability to use a complete characterisation of the distribution opens up many problem-solving opportunities, but also relies on the validity of a very comprehensive set of assumptions. This is usually considered as a drawback of this generic framework.

Our objective in this paper was to show that this apparent drawback does not constitute an obstacle in practical applications. Furthermore, the knowledge of the complete distribution actually turns out to be a highly valuable asset, as it encodes implicit information in a very synthetic way. The power of the approach has been illustrated through a rather classical application in the option pricing area. Thanks to the scenario-tree technique, this paper has shown that it is possible to extract initial information from a series of options, synthesise it using a flexible and parsimonious set of assumptions, and re-expand it again to a high degree of accuracy. On a cross-section of options data, the method proves to be extremely reliable — initial information on option prices is restored almost perfectly by the procedure — but also, and more importantly, very effective: all options that were not used to build the initial information set are priced in a very accurate way.

In terms of precision, the method compares favorably with a version of Rubinstein's algorithm (1994), slightly modified to ensure proper comparability with our scenario-tree approach. Yet, the superiority of the latter rests on its extreme parsimony and flexibility in assumptions. The direct expansion of the number of nodes in the scenario tree ensures feasibility and opens the way for any distribution of the underlying asset. In this context, our parsimonious moment-related estimation is obviously sufficient to ensure a reasonable degree of accuracy.

Beyond the particular illustration that we have selected, a central message of this paper is that practitioners should not be deterred by the high degree of complexity of the inputs associated with the scenario tree approach. When properly used, the information-preserving character of the framework allows to simultaneously achieve flexibility and accuracy, an unavoidable requirement for derivatives pricing and hedging applications. As the implementation of such an approach is indeed not an obstacle either, we strongly believe in its practical potential.

In a broader context, the scenario tree model may also prove a very useful tool to expand the set of derivatives instruments that can be simultaneously priced. By increasing the number of periods and simulating a wide range of possible sample paths of the underlying asset, one can easily adapt the framework to all kinds of exotic options and other derivative securities with the same level of confidence. A systematic exploration of such extensions is left for future research.

References


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