Who Needs Inflation Hedging?
A Quantitative Analysis of the Benefits of Inflation-Linked Bonds, Real Estate and Commodities for Long-Term Investors with Inflation-Linked Liabilities

January 2014

Lionel Martellini
Professor of Finance, EDHEC Business School
Scientific Director, EDHEC-Risk Institute

Vincent Milhau
Deputy Research Director, EDHEC-Risk Institute
Abstract
This paper proposes an empirical analysis of the opportunity gains (costs) involved in introducing (removing) various assets with attractive inflation-hedging properties for long-term investors facing inflation-linked liabilities. Using formal intertemporal spanning tests, we find that interest rate risk dominates inflation risk so dramatically within instantaneous liability risk that introducing or removing inflation-linked bonds, or real estate and commodities, from their liability-hedging portfolio has relatively little impact on investors' welfare from a short-term perspective. This holds true in spite of the attractive (in the case of real assets) and even perfect (in the case of inflation-linked bonds) inflation-hedging benefits of some of these asset classes. More substantial welfare gains/losses, comparable to the costs of ignoring equity return predictability, are obtained as the time-horizon converges towards the liability maturity date and as the relative importance of inflation risk within liability risk increases. Even more substantial utility gains are obtained if these asset classes are also used in the performance-seeking portfolio, where they provide diversification benefits with respect to equity returns.

This research has benefitted from the support of the “Advanced Investment Solutions for Liability Hedging for Inflation Risk” research chair, supported by Ontario Teachers' Pension Plan. We are grateful to Bernd Scherer for useful comments, and to Andrea Tarelli for helpful research assistance.

EDHEC is one of the top five business schools in France. Its reputation is built on the high quality of its faculty and the privileged relationship with professionals that the school has cultivated since its establishment in 1906. EDHEC Business School has decided to draw on its extensive knowledge of the professional environment and has therefore focused its research on themes that satisfy the needs of professionals.

EDHEC pursues an active research policy in the field of finance. EDHEC-Risk Institute carries out numerous research programmes in the areas of asset allocation and risk management in both the traditional and alternative investment universes.
1. Introduction
A recent surge in inflation uncertainty has increased the need for investors to hedge against unexpected changes in price levels. Inflation hedging is a concern of particularly critical importance for pension funds, in situations when pension payments are indexed with respect to consumer price or wage level indexes. The implementation of inflation-hedging portfolios has become relatively straightforward in specific contexts where either cash instruments (inflation-linked bonds, such as Treasury inflation protected securities, or TIPS) or dedicated OTC derivatives (such as inflation swaps) can be used to achieve effective inflation hedging. More generally, however, the lack of capacity for inflation-linked cash instruments and the increasing concern over counterparty risk for derivatives-based solutions leave most investors with the presence of non-hedgeable inflation risk. In this context, a variety of financial asset classes (stocks and bonds) and real asset classes (commodities or real estate in particular) have been analysed in terms of their ability to provide attractive inflation-hedging benefits. The results of such empirical investigations, however, have been mixed.

On the one hand, starting with traditional asset classes, stock investments appear as relatively poor inflation hedging vehicles from a short-term perspective. Empirical evidence suggests that there is in fact a negative relationship between expected stock returns and expected inflation (see Fama and Schwert (1977), Gultekin (1983) and Kaul (1987) among others), which is consistent with the intuition that higher inflation leads to lower economic activity, thus depressing stock returns (e.g. Fama (1981)). On the other hand, higher future inflation leads to higher dividends and thus higher returns on stocks (Campbell and Shiller (1988)), and thus equity investments should offer significant inflation protection over longer horizons, a fact that has been confirmed by a number of recent empirical academic studies (Boudoukh and Richardson (1993) or Schotman and Schweitzer (2000)). This property is a priori particularly appealing for long-term investors such as pension funds, who need to match increases in price level at the horizon, but not necessarily on a monthly basis. Similar inflation hedging properties are expected for bond returns. Indeed, bond yields may be decomposed into a real yield and an expected inflation components. Since expected and realised inflation move together on the long-term (see Schotman and Schweitzer (2000)), we expect a positive long-term correlation between bond returns and changes in inflation. In the short-term, however, expected inflation may deviate from the actual realised inflation, leading to low or negative correlations. There again, an investor willing and able to relax short-term constraints to focus on long-term inflation hedging properties may find that investing in nominal bonds can provide a useful complement to investing in inflation-linked securities.

Moving beyond traditional investment vehicles such as stocks and bonds, recent academic research has also suggested that alternative forms of investments may offer attractive inflation-hedging benefits. Commodity prices, in particular, have been found to be leading indicators of inflation in that they are quick to respond to economy-wide shocks to demand. Commodity prices generally are set in highly competitive auction markets and consequently tend to be more flexible than prices overall. Beside, recent inflation is heavily driven by the increase in commodity prices, in particular in the domain of agriculture, minerals and energy. Consistent with these theoretical arguments, a recent study by Gorton and Rouwenhorst (2006) find that, over the 1959-2004 period, commodity futures were positively correlated with inflation, unexpected inflation, and changes in expected inflation. They also find that inflation correlations tend to increase with the holding period and are larger at return intervals of 1 and 5 years than at the monthly or quarterly frequency. In the same spirit, it has also been found that commercial and residential real estate provide at least a partial hedge against inflation, which implies that portfolios that include real estate allow for enhanced inflation hedging benefits (see Fama and Schwert (1977), Hartzell et al. (1987) or Rubens et al. (1989)). This effect seems to be particularly significant over long-horizons. Hence, Anari and Kolari (2002) examine the long-run impact of inflation on homeowner equity by investigating the relationship between house prices and the prices of
non housing goods and services, rather than return series and inflation rates, and infer that house prices are a stable inflation hedge in the long-run. The implications of such findings for asset-liability management (ALM) have been discussed by Hoevenaars et al. (2008), who construct optimal mean-variance portfolios with respect to inflation-driven liabilities, using the vector-autoregressive (VAR) approach from the literature on return predictability (see Kandel and Stambaugh (1996), Campbell and Viceira (1999) and Barberis (2000), among others).

What justifies the interest in inflation-hedging properties of various asset classes is the intuition that investors (pension funds in particular) with inflation-linked liabilities need to invest in assets that are positively correlated with inflation. In this paper, we argue that this seemingly straightforward intuition is wrong, or at least severely incomplete. To see this, one may simply recognise that what matters is not the inflation-hedging properties of various asset classes, but instead their liability-hedging properties, with the two concepts coinciding only at liability maturity. For example, while inflation-linked bonds can perfectly hedge inflation-linked cash flows at any given maturity, they exhibit low short-term correlation with changes in inflation. In this context, our paper proposes a quantitative analysis of the opportunity costs (gains) involved in removing (introducing) different financial and real assets with various inflation-hedging properties within the context of a formal intertemporal portfolio selection problem. To achieve this objective, we first solve for the optimal allocation of a long-term investor facing inflation-linked liabilities in the presence of interest rate risk, inflation risk, as well as unspanned equity risk premium uncertainty. We also provide a quasi-analytical characterisation (up to the solution of a system of ordinary differential equations) for the expected utility obtained with a class of allocation strategies such that the volatility vector of wealth is linear in the stochastic equity premium state variable. In particular, such strategies include strategies where the allocation to any given asset in the asset mix is constrained to be zero, thus allowing for a straightforward analysis of the welfare cost involved in removing one particular asset class from the asset mix. We then perform an empirical calibration of the model using a parsimonious sequential maximum likelihood procedure. Using the calibrated parameter values, we finally analyse the fraction of liability risk that is spanned by various portfolios that include, or not, assets with attractive inflation-hedging properties, and we compute the term structure of welfare gains associated with these additions or removals. Focusing on welfare gains at various horizons is important because long-term investors, who by definition have preferences expressed over the value of their portfolio at a long horizon, also show substantial risk-aversion with respect to short-term uncertainty in their funding ratio. This focus on short-term risk can be induced by the presence of formal minimum funding ratio regulatory constraints, or more generally by the short-term concern expressed by various stakeholders such as trustees, beneficiaries, plan sponsors, pension fund managers, etc.

From the short-term perspective, our results first show that inflation risk should not be regarded as a quantitatively important problem for investors endowed with inflation-linked liabilities. For example, for our calibrated set of parameter values, we find that a liability-hedging portfolio solely invested in nominal bonds already spans close to 90% of the instantaneous variance of an inflation-indexed liability portfolio for a 5-year horizon. This percentage further increases to approximately 95% or 97% when the horizon is extended to 10 years or 15 years. In this context, introducing assets such as inflation-linked bonds with perfect inflation hedging properties can merely increase by a few percents (from 90%, 95% or 97% to 100%) the proportion of the variance of the liabilities that can be hedged by the liability-hedging portfolio. Obviously, the marginal improvement is even weaker when assets with imperfect inflation-hedging properties, such as real estate and commodities, are being used for inflation hedging purposes in the absence of inflation-linked bonds. While surprising at first sight, these results can be explained by a relatively straightforward intuition. Short-term (instantaneous) liability risk is driven by two main risk factors, namely interest rate risk and inflation risk, and it turns out that for reasonable

1 - A similar result has been established by Sangvinatis and Wachter (2005) for several state variables, under the assumption that all state variables were spanned by the traded assets. We extend this result to the case of unspanned equity premium risk.
parameter values, interest rate risk dominates inflation risk so substantially that introducing or removing assets with attractive inflation-hedging properties has little impact on the investors’ welfare. From a mathematical point, this domination of interest rate risk over inflation risk can be explained by the fact that interest rate risk affects instantaneous liability risk through the impact on the discount factor, an impact that increases linearly with time-horizon (see equation (2.10) for a formal argument). Inflation risk, on the other hand, affects the value of the liabilities through an impact on the cash-flows, which is not affected by time-horizon (see again equation (2.10)). In fact, it is only in the case of exceedingly short horizons (e.g., a year) that the relative importance of inflation risk versus interest rate risk becomes substantial within total liability risk.

While inflation risk hedging is not a meaningful problem from the short-term perspective, one would expect its importance to increase substantially while getting close to horizon. Indeed, when the investment horizon coincides with the liability horizon, interest rate risk gradually vanishes away to leave inflation risk remain as the sole source of uncertainty in the liability payment. To analyse these questions, we measure the welfare losses associated with removing from the liability-hedging portfolio various asset classes with attractive (real estate, commodities) or even perfect (inflation-linked bonds) inflation-hedging properties at horizon, so that the investors only has access to nominal bonds in addition to stocks for liability-hedging purposes. Nominal bonds, while extremely efficient at hedging liability risk from the short-term perspective (since they allow for a perfect hedge of interest rate risk, the dominant force from the short-term perspective) turn out to be extremely ill-suited from the long-term perspective because of their poor inflation hedging properties. For reasonable parameter values, our results show welfare losses from removing inflation-linked bonds from the asset mix that are economically significant, especially for high risk aversion levels, and of the same order of magnitude as the welfare cost involved in ignoring predictability in equity returns. These results are obtained for relatively low inflation volatility, with a value that is 1.88% in our sample period. Obviously, the importance of inflation hedging would be smaller if inflation volatility turns out to be particularly low in a given period of time, for example under the influence of a central bank that targets inflation with a tight control around a specified inflation target, and it would be higher in economic regions with more pronounced inflation uncertainty.

Beyond their usefulness, or lack thereof, for liability-hedging purposes, real assets (and inflation-linked bonds) may also be useful in the performance-seeking component of investors’ portfolios due to their positive risk premia and their relatively low correlations with stocks and bonds, which makes them attractive for diversification purposes. The utility losses of removing real assets from all building blocks, which are trivially larger than those of ignoring them only in the LHP, can reach extremely high levels for aggressive investors. These results therefore suggest that real assets provide benefits to long-term investors, even if their usefulness is limited in terms of inflation and liability hedging purposes.

Our paper is of course not the first to analyse the long-term investment problem in the presence of inflation risk. In a continuous-time model similar to ours, Brennan and Xia (2002) show by mathematical arguments that the introduction of inflation-indexed bonds increases investor’s welfare, unless the investor has too little risk aversion. In a discrete-time VAR model, Campbell et al. (2003) show that investors with infinite risk aversion and recursive utility from consumption hold only perpetual real bonds. They also document that the introduction of inflation-linked bonds in the investment universe leads to large utility gains for all investors, whatever their risk aversion. Campbell et al. (2009) argue that indexed bonds allow investors endowed with real liabilities to reduce the long-term risk of their portfolio more substantially than other asset classes would do. On a different note, they also point out that inflation-indexed bond prices provide information that are useful to infer inflation expectations, which is of interest for monetary authorities. Other papers have studied the benefits of adding real assets to a portfolio of stocks

2. We assume away longevity risk in this discussion, as in the rest of the paper.
3. “Although clues about inflation expectations abound in financial markets, inflation-indexed securities would appear to be the most direct source of information about inflation expectations and real interest rates” (Governor Ben S. Bernanke before the Investment Analysts Society of Chicago, April 15, 2004. Also quoted in Grishchenko and Huang (2008)).
and bonds. Froot (1995) defines real assets as assets “that tend to increase in nominal value in the face of inflation” and notes that commodities are good at diversifying stocks and nominal bonds. Using a discrete-time VAR, Hoevenaars et al. (2008) examine the properties of alternative classes such as commodities and real estate. They report that commodities have both attractive inflation-hedging properties and low correlation with stocks and bonds, which make them well-suited for diversifying a portfolio. They also report that adding alternative asset classes, namely real estate, commodities and hedge funds, and corporate bonds, generates utility gains for long-term investors. But they do not measure the gains for separate asset classes. Moreover, these gains are based on the assumption that the investor follows a fixed-mix strategy, which is not optimal when the opportunity set is time-varying.

We complete these papers in several directions. First, our analysis is done in a continuous-time model with stochastic investment opportunities which allows for an analytical derivation of the optimal strategy. This model and its solution are not new (see e.g. Munk et al. (2004)), but we use them to derive term structures of inflation and liability-hedging qualities. It is known that continuous-time models admit a VAR representation in discrete time (see Campbell et al. (2004)), but they impose no-arbitrage restrictions that are not present in unrestricted VAR models. Moreover, a traditional criticism of VAR models is that their coefficients lack interpretability. In contrast, the parameters of a continuous-time model have interpretations. Second, we provide for the first time, based on reasonable parameter values calibrated to Canadian and US data, a quantitative estimate of the welfare loss induced by removing inflation-indexed bonds from the liability-hedging portfolio, as opposed to analysing the question from an asset-only perspective. This allows us to perform an independent analysis of the benefits of inflation-linked bonds, commodities or real estate for liability hedging purposes versus diversification purposes. The computation of welfare gains is based on the strategy that is optimal in the presence of the stochastic opportunity set, so that we do not have to assume a fixed-mix. Third, we assess the benefits of introducing real assets such as real estate and commodities, which have attractive but imperfect, inflation-hedging properties. We also provide a detailed analysis of the term structure of welfare gains, which sheds new light on the conflicting results obtained when addressing the question of liability or inflation hedging from a short-term or a long-term perspective.

The rest of the paper is organised as follows. In Section 2, we introduce the model. Section 3 is dedicated to the empirical calibration of the model. In section 4 we focus on the relevance of inflation risk by analysing the welfare losses related to the removal of inflation-linked bonds from liability-hedging portfolios. In Section 5, we extend the analysis to real estate and commodities, which are found to have attractive, albeit imperfect, inflation hedging properties. Section 6 concludes and proposes a number of suggestions for further research. Technical details are relegated to a dedicated appendix.

2. The Model
In this section, we present the models we use for risk factors impacting asset and liability values, and we solve for the utility-maximising portfolio strategies from the perspective of an investor with preferences over wealth relative to inflation-linked liabilities.

2.1 Financial Variables
Uncertainty in the economy is modeled by a standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\). \(\mathbb{P}\) is a probability measure defined on \((\Omega, \mathcal{F})\); it represents the assessment by the representative investor of the probabilities of measurable events, i.e. the elements of the sigma-field \(\mathcal{F}\). To take into account the dynamic aspect of portfolio decisions, we also introduce a finite time span \([0, \tau]\), as well as a filtration \((\mathcal{F}_t)_{0 \leq t \leq \tau}\). All processes considered in this paper will be assumed to be progressively measurable with respect to this filtration.
The first state variable is the nominal short-term rate $r$, which we assume to evolve as (see Vasicek (1977)):

$$
\frac{dr_t}{r_t} = a(b - r_t) \, dt + \sigma^r \, dz^r_t,  \tag{2.1}
$$

where $z^r$ is a Brownian motion.

The nominal rate is the logarithmic rate of return (expressed in annual terms) on a risk-free asset (a bank account, which will be loosely referred to as "cash" in the paper) over an infinitesimal period $[t, t + dt]$. The value of the bank account at any date is therefore:

$$
S^0_t = \exp \left( \int_0^t r_s \, ds \right).
$$

The model (2.1) implies that if there exists a constant market price of interest rate risk, $\lambda^r$, then the nominal zero-coupon yields of maturity $\tau$ at date $t$ is (see Vasicek (1977) and the many subsequent papers on bond pricing in affine models of the term structure):

$$
y(t, \tau) = \frac{D(\tau)}{\tau} i_t - \frac{\tau}{\tau},  \tag{2.2}
$$

where

$$
D(\tau) = \frac{1 - e^{-a\tau}}{a}, \quad C(\tau) = \bar{b}[D(\tau) - \tau] + \frac{\sigma^r}{2a} \left[ \tau - 2D(\tau) + \frac{1 - e^{-2a\tau}}{2a} \right],
$$

$$
\bar{b} = b - \frac{\sigma^r \lambda^r}{a}.  \tag{2.3}
$$

The investor has access to a constant-maturity bond index with maturity $\tau_B$, whose price evolves

$$
\frac{dB_t}{B_t} = [r_t - D(\tau_B)\sigma^r \lambda^r] \, dt - D(\tau_B)\sigma^r \, dz^r_t.  \tag{2.4}
$$

Since the entire yield curve is driven by a single factor, any bond with a given maturity can be replicated by dynamically mixing the constant-maturity bond and the cash. We will take $\tau_B = 10$ years in our empirical exercise.

In addition to the bond index, the investor can also trade in a stock index $S$ with stochastic Sharpe ratio:

$$
\frac{dS_t}{S_t} = [r_t + \sigma^S \lambda^S] \, dt + \sigma^S \, dz^S_t.  \tag{2.5}
$$

Dividends are assumed to be continuously reinvested in the index. As in Kim and Omberg (1996), the Sharpe ratio is assumed to follow a mean-reverting process:

$$
\frac{d\lambda^S_t}{\lambda^S_t} = \kappa (\bar{\lambda} - \lambda^S_t) \, dt + \sigma^\lambda \, dz^\lambda_t.  \tag{2.6}
$$

Empirical calibrations of this model typically yield strongly negative values for the correlation $\rho^S$ (see Barberis (2000), Xia (2001) and our own calibration in Section 3). This finding justifies, to some extent, the assumption that this parameter is equal to $-1$. For the sake of generality, our analytical and numerical results are derived in a more general framework, that allows for a non-perfect correlation between the two processes.

There also exists a real asset $Y$, whose value evolves as:

$$
\frac{dY_t}{Y_t} = [r_t + \sigma^Y \lambda^Y] \, dt + \sigma^Y \, dz^Y_t.  \tag{2.7}
$$
In empirical applications this real asset will be interpreted either as a real estate index or as a commodity index.

The price index $\Phi$ evolves as a Geometric Brownian motion:

$$\frac{d\Phi_t}{\Phi_t} = \pi \, dt + \sigma^\Phi \, dz_t^\Phi. \tag{2.7}$$

We summarise all payments due by the pension fund to its beneficiaries in a single payment at date $\tau_L$. In practice, $\tau_L$ can be taken equal to the duration of pension liabilities, and we will let this parameter vary in our empirical exercise. Liabilities are therefore represented by an inflation-indexed zero-coupon bond with fixed maturity date $\tau_L$, that is a zero-coupon that pays $\Phi_{\tau_L}$ at date $\tau_L$. Modelling liabilities as an inflation-indexed bond is standard in the related literature (see e.g. Hoevenaars et al. (2008) and Detemple and Rindisbacher (2008)). The choice of a zero-coupon bond is motivated by the desire to obtain explicit expressions for the optimal portfolio policy, which is usually not possible with coupon bonds, due to the stochastic volatility of the value of the liability value. Finally, we choose a zero-coupon with a vanishing time-to-maturity, not a constant one, because in a constant-maturity bond, the exposure to interest rate risk would be constant, and as will be argued later, it would be of greater magnitude than the exposure to inflation risk. Instead, liabilities with vanishing time-to-maturity have a decreasing exposure to interest rate risk, and are therefore more exposed to inflation risk near maturity. This discussion may seem somewhat obscure at this stage, but it should become clearer in Section 4.

If the pension fund is unable to hedge perfectly inflation risk, then it operates in an incomplete market, so there are infinitely many possible prices for inflation risk, and each price for inflation risk gives a price for the inflation-indexed bond that is comprised between the no-arbitrage bounds (see El Karoui and Quenez (1995)). Among all possible prices for inflation risk are constant prices. In this paper, we assume that a constant price $\lambda^\Phi$ is given. From Martellini and Milhau (2010), the price of an indexed zero-coupon bond maturing at date $\tau_L$ is thus:

$$I(t, \tau_L) = E_t \left[ \frac{M_{\tau_L}}{M_t} \Phi_{\tau_L} \right] = \Phi_t \exp \left[-D(\tau_L-t)\pi_t + E(\tau_L-t)\right], \quad t \leq \tau_L, \tag{2.8}$$

with:

$$E(u) = \left( b - \sigma^r \lambda^r \right) [D(u) - u] + \left( \pi^r - \sigma^\Phi \lambda^\Phi \right) u - \frac{\sigma^r \sigma^\Phi \rho}{a} \left[ u - \frac{1 - e^{-au}}{a} \right] + \frac{(\sigma^r)^2}{2a^2} \left[ u - 2 \frac{1 - e^{-au}}{a} + \frac{1 - e^{-2au}}{2a} \right].$$

For notational brevity, we will denote with $L$ the price of the payoff $\Phi_{\tau_L}$ that is obtained with the constant $\lambda^\Phi$:

$$L_t = I(t, \tau_L),$$

and we will refer to $L$ as the value of liabilities, or the liability process. We emphasise that $L_t$ is not the unique possible price for the payoff $\Phi_{\tau_L}$. Other price processes can be obtained by choosing an arbitrary price of inflation risk, not necessarily constant over time. Our choice to work with the process $L$ instead of another price for liabilities implies that the pension fund takes $L$ as a benchmark.

A straightforward application of Ito’s lemma to (2.8) shows that the dynamics of $L$ is:

$$\frac{dL_t}{L_t} = \left[ \pi_t - D(\tau_L-t)\sigma^r \lambda^r + \sigma^\Phi \lambda^\Phi \right] \, dt + D(\tau_L-t)\sigma^r \, dz_t^\Phi + \sigma^\Phi \, dz_t^\Phi. \tag{2.9}$$

5 - Some related papers have considered a stochastic expected inflation rate (see e.g. Brennan and Xia (2002) and Munk et al. (2004)).

6 - In practice, pension funds in a stationary state have a constant-maturity liability and pay continuously inflation-indexed coupons. This situation can be represented by the sum of a constant-maturity indexed zero-coupon and a perpetual indexed bond that pay coupons. Such a model would be, however, far less tractable than the model adopted in this paper, because the liability process, which would be the sum of the prices of these bonds, would have stochastic volatility.

7 - An alternative price is the one computed under the minimal martingale measure, that assigns a zero price to non-traded risks, and is obtained by taking $\xi = 0$ in (2.11). Another example is the price computed under the minimax martingale measure, defined by the minimax state-price deflator given in proposition 1.
We shall sometimes assume that the investor has access to an inflation-indexed bond with a constant maturity \( \tau_0 \) of ten years. As shown in appendix A.1, the dynamics of this bond is:

\[
\frac{dI_t}{I_t} = \left[ r_t - D(\tau_1)\sigma^r \lambda^r + \sigma^\phi \lambda^\phi \right] dt - D(\tau_1)\sigma^r dz^r_t + \sigma^\phi dz^\phi_t.
\]

In this case, inflation risk is spanned, so \( L \) is the unique no-arbitrage price for the payoff \( \Phi_{\tau_t} \).

All previous dynamics can be rewritten in vector form, making use of a 5-dimensional Brownian motion \( z \) and volatility vectors:

\[
\begin{align*}
\frac{dr_t}{r_t} &= \alpha (b - r_t) dt + \left( \sigma^r \right)' dz_t, \\
\frac{dB_t}{B_t} &= \left[ r_t - D(\tau_B)\sigma^r \lambda^r \right] dt + \sigma^B(\tau_B)' dz_t, \\
\frac{dS_t}{S_t} &= \left[ r_t + \sigma^S \lambda^S_t \right] dt + \left( \sigma^S \right)' dz_t, \\
d\lambda^S_t &= \kappa (\lambda^S_t - \lambda^S_t) dt + \left( \sigma^\lambda \right)' dz_t, \\
\frac{d\Phi_t}{\Phi_t} &= \pi dt + \left( \sigma^\phi \right)' dz_t, \\
\frac{dY_t}{Y_t} &= \left[ r_t + \sigma^Y \lambda^Y \right] dt + \left( \sigma^Y \right)' dz_t, \\
\frac{dL_t}{L_t} &= \left[ r_t - D(\tau_L - t)\sigma^r \lambda^r + \sigma^\phi \lambda^\phi \right] dt + \sigma^L(\tau_L - t)' dz_t, \\
\frac{dI_t}{I_t} &= \left[ r_t - D(\tau_I)\sigma^r \lambda^r + \sigma^\phi \lambda^\phi \right] dt + \sigma^I(\tau_I)' dz_t.
\end{align*}
\]

The volatility vectors of the constant-maturity bonds and of the liability value are given by:

\[
\begin{align*}
\sigma^B(\tau_B) &= -D(\tau_B)\sigma^r, & \sigma^I(\tau_I) &= -D(\tau_I)\sigma^r + \sigma^\phi, \\
\sigma^L(\tau_L - t) &= -D(\tau_L - t)\sigma^r + \sigma^\phi.
\end{align*}
\]

The volatility matrix of the traded assets collects the volatility vectors of all traded assets (except cash which is locally risk-free). In this paper, we shall consider three main investment universes, which lead to different expressions for the volatility matrix:

1. The traded assets are bonds and stocks only. Then:

\[
\sigma = \begin{pmatrix} \sigma^B(\tau_B) & \sigma^S \end{pmatrix}
\]

A price of risk vector is defined by:

\[
\lambda_t = \sigma(\sigma')^{-1} \begin{pmatrix} \sigma^r \lambda^r \\ \sigma^S \lambda^S_t \end{pmatrix}.
\]

2. The traded assets are bonds, stocks and the real asset. Then:

\[
\sigma = \begin{pmatrix} \sigma^B(\tau_B) & \sigma^S & \sigma^Y \end{pmatrix}
\]

and:

\[
\lambda_t = \sigma(\sigma')^{-1} \begin{pmatrix} \sigma^r \lambda^r \\ \sigma^S \lambda^S_t \\ \sigma^Y \lambda^Y \end{pmatrix}.
\]
3. The traded assets are bonds, stocks and the inflation-indexed bond $I$. Then:

$$\sigma = \begin{pmatrix} \sigma^B(\tau_B) & \sigma^S & \sigma^I(\tau_I) \end{pmatrix},$$

and:

$$\lambda_t = \sigma(\sigma')^{-1} \begin{pmatrix} \sigma'^\tau \lambda' \\ \sigma^S\lambda_t^S \\ -D(\tau_I)\sigma'^\tau + \sigma^\Phi\lambda^\Phi \end{pmatrix}. $$

Other investment universes can be obtained by combining nominal bonds and the real asset only, or by considering one asset class in isolation from the others. Such examples will be given in the text. In each case, we can represent the price of risk vector as an affine function of the Sharpe ratio of the stock:

$$\lambda_t = \Lambda^1 + \Lambda^2, \quad t \geq 0.$$

In this incomplete market setting, where incompleteness is driven by unspanned equity risk premium uncertainty (when $|\rho^{SA}| < 1$) as well as inflation risk (when the inflation-linked bond is not part of the menu of traded assets), an infinite number of state-price deflators, or equivalent martingale measures exist. Each of them is associated with price of risk vectors that can be written as $\lambda + \nu$, where $\sigma'\nu_t = 0$ almost surely for all $t$ (see He and Pearson (1991)). $\lambda$ is the only one that is spanned by the volatility matrix of traded assets, which makes it particularly interesting in subsequent computations. The state-price deflator obtained with the price of risk vector $\lambda + \nu$ is:

$$M_t = \exp \left[ -\int_0^t \left( r_s + \frac{\|\lambda_s\|^2 + \|\nu_s\|^2}{2} \right) \, ds - \int_0^t (\lambda_s + \nu_s)' \, dz_s \right], \quad t \geq 0. \quad (2.11)$$

### 2.2 Optimal Portfolio Choice

We now solve for the optimal portfolio policy for an investor facing the opportunity set described in Section 2. The portfolio is self-financed, so the strategy is completely characterised by the vector $w_t$ that collects the weights allocated to the locally risky assets at time $t$. We let $dR_t$ denote the vector of instantaneous arithmetic returns on these assets. In case 1, we have:

$$dR_t = \begin{pmatrix} \frac{dB_t}{B_t} & \frac{dS_t}{S_t} \end{pmatrix}' ,$$

while in case 2:

$$dR_t = \begin{pmatrix} \frac{dB_t}{B_t} & \frac{dS_t}{S_t} & \frac{dY_t}{Y_t} \end{pmatrix}' ,$$

and in case 3:

$$dR_t = \begin{pmatrix} \frac{dB_t}{B_t} & \frac{dS_t}{S_t} & \frac{dI_t}{I_t} \end{pmatrix}' .$$

These weights do not necessarily sum up to 1, and cash is used to make the balance. In each case, the dynamics of the wealth can be described by the following budget equation:

$$\frac{dA_t}{A_t} = w_t' dR_t + (1 - w_t' 1) \frac{dS_t^0}{S_t^0} = (r_t + w_t' \sigma_t) \, dt + w_t' \sigma' \, dz_t. $$

The investor has horizon $T$, which is assumed to be such that:

$$T \leq \tau_L.$$
With these specifications, the portfolio choice problem can be written as the problem of maximising expected utility from terminal real wealth:

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad x > 0.$$  

The horizon $T$ is assumed to be less than, or equal to the maturity of liabilities, $\tau_L$. For $T < \tau_L$, the problem (2.12) would be different if we had chosen a price process for liabilities different from $L$. For $T = \tau_L$, the choice of a particular price process is irrelevant, because all possible prices for liabilities are equal at date $\tau_L$.

The problem (2.12) is close to that of maximising expected utility from terminal real wealth, and is even strictly equivalent to it when the investment horizon coincides with the maturity of liabilities ($T = \tau_L$). The problem of maximising utility from real wealth has been solved in several recent papers such as Brennan and Xia (2002), Munk et al. (2004) and Sangvinatsos and Wachter (2005). We follow these authors by solving (2.12) via the martingale approach. It is based on the equivalence between the "dynamic problem" (2.12) and a "static problem" where the control variable is the terminal wealth. The only concern is that in cases 1 and 2, the market is incomplete, because inflation risk is not necessarily spanned by the bond, the stock and the real asset. Only in case 3 is the market complete if we further assume that $\rho^{SL} = -1$. The martingale approach in incomplete markets, developed by He and Pearson (1991), can be applied to such problems. The method proceeds as follows. First, a candidate optimal terminal wealth is computed by solving the following "static" portfolio choice problem:

$$\max_{X} \mathbb{E} \left[ U \left( \frac{X}{L_T} \right) \right], \quad \text{subject to} \quad \mathbb{E} [M_T X] = A_0,$$  

for a generic state-price deflator $M$ of the form (2.11). The candidate optimal payoff may not be replicable due to market incompleteness. But there is one state-price deflator, or equivalently one choice for $\nu$, that leads to a replicable payoff. This payoff is the optimal terminal wealth, and the corresponding state-price deflator is called "minimax", following the terminology coined by He and Pearson. The last step in the solution is to find the portfolio strategy that replicates the optimal terminal wealth. This strategy is the solution to the original dynamic problem (2.12). The following proposition, the proof of which is given in appendix A.2, describes this solution.

**Proposition 1**

- **The optimal wealth process in (2.12) is:**

$$A_t^* = \frac{A_0}{\mathbb{E} \left[ (M_T^*)^{-\frac{1}{\gamma}} L_T^{-\frac{1}{\gamma}} g(t, \lambda_t^S) \right]} \left( M_T^* \right)^{-\frac{1}{2}} L_T^{\frac{1}{2}} g(t, \lambda_t^S),$$

- **The optimal portfolio rule is given by:**

$$\nu_t^* = (1 - \gamma)N \sigma^L (\tau_L - t) - \gamma \frac{g_\lambda}{g} N \sigma^\lambda,$$

$$g(t, \lambda_t^S) = \exp \left[ \frac{1 - \gamma}{\gamma} \left( C_1 (T - t) + C_2 (T - t) \lambda_t^S + \frac{1}{2} C_3 (T - t) (\lambda_t^S)^2 \right) \right].$$

- **The optimal portfolio rule is given by:**

$$w_t^* = \frac{\lambda_t^{PSP}}{\gamma \sigma_t^{PSP}} \frac{w_t^{PSP}}{\gamma \sigma_t^{PSP}} \beta_t^L w_t^{LHP} - \left( 1 - \frac{1}{\gamma} \right) \left[ C_2 (T - t) + C_3 (T - t) \lambda_t^S \right] \beta_t^L w_t^{LHP},$$  

(2.14)
where:

\[ w^\text{PSP}_t = \frac{1}{1' (\sigma' \sigma)^{-1} \sigma' \lambda_t} (\sigma' \sigma)^{-1} \sigma' \lambda_t, \]

\[ w^\text{LHP}_t = \frac{1}{1' (\sigma' \sigma)^{-1} \sigma' \sigma^L (\tau_L - t)} (\sigma' \sigma)^{-1} \sigma' \sigma^L (\tau_L - t), \]

\[ w^\lambda = \frac{1}{1' (\sigma' \sigma)^{-1} \sigma' \lambda} (\sigma' \sigma)^{-1} \sigma' \lambda, \]

\[ \frac{\lambda^\text{PSP}}{\sigma^\text{PSP}_t} = 1' (\sigma' \sigma)^{-1} \sigma' \lambda_t, \quad \beta^\text{L}_t = 1' (\sigma' \sigma)^{-1} \sigma' \sigma^L (\tau_L - t), \quad \beta^\lambda = 1' (\sigma' \sigma)^{-1} \sigma' \sigma^L. \]

- \( N \) is the matrix of the projection onto the orthogonal space of the columns of \( \sigma \):

\[ N = I - \sigma (\sigma' \sigma)^{-1} \sigma', \]

and the functions \( C_1, C_2 \) and \( C_3 \) are solutions to ordinary differential equations (ODEs) that are given in appendix A.2.

Appendix A.2 also shows how to express the indirect utility function, i.e., the expected utility from the terminal funding ratio when the optimal strategy is taken. We have:

\[ J \left( t, \frac{A_t}{L_t}, \lambda^S_t \right) = U \left( \frac{A_t}{L_t} \right) g(t, \lambda^S_t), \]

where \( g(t, \lambda^S_t) \) is given in proposition 1.

The decomposition (2.14) is standard, in view of the results of Detemple and Rindisbacher (2010), who establish a general separation formula for optimal portfolios, and of those of Munk et al. (2004), who study a model close to ours. The first building block is the performance-seeking portfolio (PSP), which has the same structure as in the solution to the model with constant investment opportunities (see Merton (1969)). Here, the composition of this portfolio has to be continuously revised in response to changes in market conditions, which are summarised here by the Sharpe ratio of the equity index. The allocation to the PSP is an increasing function of the Sharpe ratio of this portfolio, and a decreasing function of its volatility and of the investor’s risk aversion. The positive demand for the PSP arises because the investor is concerned with the risk-return trade-off of his portfolio over the next trading interval: being the portfolio that achieves the highest Sharpe ratio, the PSP is relevant from this perspective.

The second building block can be regarded as a liability-hedging portfolio (LHP). By definition, the LHP maximises the squared correlation between innovations to the liability value \( L \) and the innovations to a portfolio of traded assets. This maximum squared correlation would be equal to 1 if liability risk was spanned by the traded assets, i.e., if inflation risk was spanned. But in general it is lower than 1, because of the presence of some unspanned inflation risk. The allocation to the LHP is a function of the parameter \( \beta^L \), which is the beta of the liability value with respect to the LHP: in particular, it is increasing in the correlation between the LHP and liabilities, and it is decreasing in the volatility of the LHP. The demand for the LHP is motivated by the fact that the investor is concerned with changes in the funding ratio over each trading interval and wants to minimise the volatility of such changes. Formally, it can be shown that the variance of the change in the logarithm of \( (A/L) \) over the next interval is minimal for the following portfolio rule: \( w_t = \beta^L_t w_t^\text{LHP} \), which corresponds to an investment in the LHP and cash. The investor will allocate more to this strategy as his risk aversion grows. It can be noted that a nominal zero-coupon bond with maturity matching investor’s horizon would replace the LHP if the investor was maximising expected utility from terminal nominal wealth rather than from terminal funding ratio (see Detemple and Rindisbacher (2010)).

Finally, the third building block in (2.14) involves the portfolio \( w^\lambda \), which best hedges unexpected changes in the Sharpe ratio of the stock index. Its formal definition is similar to that of the LHP: it
maximises the squared correlation between the innovations to the Sharpe ratio and the innovations to a portfolio of traded assets. This intertemporal hedging demand term appears because equity premium is stochastic (see for example Merton (1973) and Kim and Omberg (1996)), which induces a non-trivial dependency of the optimal portfolio strategy in the time-horizon. The allocation to the portfolio $w^\lambda$ is a function of the parameter $\beta\lambda$, which is the beta of the Sharpe ratio with respect to the hedging portfolio. We note that both the hedging demand against liability risk and the hedging demand against equity premium risk are zero when the risk aversion coefficient $\gamma$ is equal to 1 (case of the log investor).

2.3 Expected Utility From Suboptimal Strategies
In the empirical Sections 4 and 5 of this paper, we will compute the expected utility associated with various suboptimal strategies, including strategies involving the removal of one particular asset class from the menu of traded assets. As shown in proposition 2.15 below, this computation can be done analytically, up to the solution of a system of ODEs, for a family of strategies such that the volatility vector of wealth is linear in the state variable $\lambda^S$:

$$\sigma w_t = H_1(T - t) + \lambda^S t H_2(T - t).$$ (2.15)

The coefficients of this combination, which are the vectors $H_1$ and $H_2$, are functions of time-to-horizon $T - t$. The utility-maximising strategy falls into this category, with:

$$H_1(u) = \frac{1}{\gamma} \frac{\sigma(\lambda^S)}{\gamma^\lambda} + \left(1 - \frac{1}{\lambda^S}\right) \sigma(\lambda^S) \frac{\sigma(\lambda^S)}{\gamma^\lambda} + \left(1 - \frac{1}{\lambda^S}\right) \beta^\lambda C_2(u) \sigma^\lambda,$$

$$H_2(u) = \frac{1}{\gamma} \frac{\sigma(\lambda^S)}{\gamma^\lambda} - \left(1 - \frac{1}{\lambda^S}\right) \beta^\lambda C_3(u) \sigma^\lambda.$$

The next proposition shows that the expected utility from such “linear” strategies is an exponential quadratic function of $\lambda^S$. A similar result has been established by Sangvinatsos and Wachter (2005) for several state variables, under the assumption that all state variables are spanned by the traded assets. The following proposition slightly extends this result by allowing for unspanned equity premium risk.8

Proposition 2
Assume that the vector of weights is of the form (2.15). Then the expected utility from terminal funding ratio is given by:

$$E_t \left[ U \left( \frac{A_T}{L_T} \right) \right] = U \left( \frac{A_T}{L_t} \right) h(t, \lambda^S)^\gamma,$$

where:

$$h(t, \lambda^S) = \exp \left[ \frac{1 - \gamma}{\gamma} \left( E_1(T - t) + E_2(T - t) \lambda^S + \frac{1}{2} E_3(T - t) \lambda^S \right)^2 \right],$$

and the functions $E_1$, $E_2$ and $E_3$ are solutions to ODEs given in appendix A.3.

The proof of this proposition is given in appendix A.3. The monetary utility loss (MUL) of some strategy A with respect to some strategy B is defined as the excess initial capital that must be invested in this strategy in order to achieve the same welfare level as with the optimal policy. To write this definition formally, let us denote with $x$ the MUL, with $EU_A(A_0)$ the expected utility from strategy A starting from the initial capital $A_0$, and with $EU_B(A_0)$ the expected utility from strategy B starting from the same initial wealth. By definition, we have that:

$$EU_A(A_0 + x) = EU_B(A_0).$$
We denote with $h^A$ and $h^B$ the functions $h$ that enter into the expected utilities of strategies A and B. It follows from proposition 2 that the MUL can be computed in quasi-closed form, and is given by:

$$\frac{x}{A_0} = \left[ \frac{h^A(0, A_0)}{h^B(0, A_0)} \right]^{\gamma-1} - 1.$$

In particular, the relative MUL, which is the ratio of the MUL over the capital $A_0$ invested in the optimal strategy, is independent from $A_0$. We can therefore normalise $A_0$ to 100 in our subsequent numerical computations.

3. Model Calibration
This section describes the calibration of the model to empirical data. Since the numerical illustrations will relate to a Canadian pension fund, we use data on Canadian inflation and term structure.

3.1 Dataset
Our dataset consists of Canadian and US quarterly data which have been extracted from Bloomberg and CRSP. We use the yields on Canadian government bonds of maturities 3 months, 1 year, 3 years, 5 years and 10 years. This span is comparable to what is used in Duffee (2002), Munk et al. (2004) and Sangvinatsos and Wachter (2005). These yields, which are extracted from Bloomberg, are available from Q1.1986 to Q1.2011. The Canadian Consumer Price Index is available over the period Q1.1956 to Q1.2011.

In order to avoid introducing home bias in the portfolio of the pension fund, we use the S&P500 index as opposed to a purely Canadian index (such as the S&P/TSX index). The ex-dividend return and total return on the S&P500 are taken from CRSP and are available over the period Q4.1964 to Q4.2010. These returns are based on the US dollar value of the index, but we need to convert US dollar values to Canadian dollar values from the perspective of the Canadian pension fund. Data on US versus Canadian dollar exchange rates is extracted from Datastream. Combining the two series of returns, we compute the dividend yield as:

$$DY_{t_i} = \log \frac{1}{S_{t_i}^{\text{ex}}} \sum_{j=i-4}^{i-1} D_{t_j},$$

where the dividend paid at date $t_j$ can be obtained as:

$$D_{t_j} = S_{t_{j-1}} \left[ \frac{S_{t_j}}{S_{t_{j-1}}} - \frac{S_{t_j}^{\text{ex}}}{S_{t_{j-1}}^{\text{ex}}} \right].$$

In this equation, $S^{\text{ex}}$ denotes the price index and $S$ is the total return index. We consider real estate and commodities as the two real assets. The real estate index is the National Council of Real Estate Investment Fiduciaries Property Index (NCREIF). It is a US index, which is available from Datastream from Q1.1978 to Q1.2011. It is non-REITS, which implies that it is not polluted by artificial exposure to the stock market. Commodities are represented by the S&P Goldman Sachs Commodity Index (GSCI), available from Q1.1970 to Q1.2011. Both the real estate and the commodities index are originally provided in US dollars, and are converted to Canadian dollars for calibration purposes.

3.2 Estimation Technique
We use a Likelihood Maximisation technique, which is similar to the one employed by Sangvinatsos and Wachter (2005). We have six state variables: the nominal short-term rate $r$, the price
index $\Phi$, the stock price $S$, its Sharpe ratio $\lambda^S$ and the real estate index $Y$. We define accordingly a state vector as:

$$V_t = \left( r_t \quad \log \Phi_t \quad \log S_t \quad \lambda^S_t \quad \log Y_t \right)' \quad (3.1)$$

Variables $\Phi$, $S$ and $Y$ are readily observable, but $r$ and $\lambda^S$ are not. The short-term rate can be inferred from observed zero-coupon yields (equation (2.2)) if one can find one observed yield that is equal to the model-implied yield. In Duffee (2002) and Sangvinatsos and Wachter (2005), three yields (of maturities 6 months, 2 years and 10 years) are assumed to be equal to model-implied yields. These authors make this assumption because they estimate three-factor models, and have therefore three factors to extract from yields at each date. Here we are considering a simpler one-factor model, so it is sufficient to assume that only one yield is equal to the model-implied yield. We choose this yield to be the 3M one, because it is the shortest maturity in our dataset. Because the model cannot fit exactly more than one yield at all dates, any other observed yield is equal to the model-implied yield plus a residual. Hence the other yields that we include in the estimation (1 year, 3 years, 5 years and 10 years) are of the form:

$$\tilde{y}(t, \tau) = y(t, \tau) + \eta^\tau_t,$$

where we assume that the residuals $\eta^\tau_t$ are both cross-sectionally and serially uncorrelated, independent from other state variables, and normally distributed with mean zero. We assume heteroskedastic residuals, so the variance $s^2_\tau = \mathbb{V}[\eta^\tau_t]$ is allowed to depend on $\tau$.

Regarding the Sharpe ratio $\lambda^S$, we follow Campbell and Viceira (1999) in assuming that the expected excess return on the stock $S$ is an affine function of the dividend yield. That is, if $\Delta t$ is our sampling period, the expected excess return of holding one share of the stock over the risk-free return on one Treasury bill of maturity $\Delta t$ is:

$$\mathbb{E}_t \left[ \log \frac{S_{t+\Delta t}}{S_t} \right] - (\Delta t) y(t, \Delta t) = mDY_t + p, \quad (3.2)$$

for some constants $m$ and $p$. It is shown in appendix A.5 that equality (3.2) implies an affine relationship between $\lambda^S_t$ and the dividend yield:

$$\lambda^S_t = \tilde{m}DY_t + \tilde{p}, \quad (3.3)$$

where expressions for $\tilde{m}$ and $\tilde{p}$ are given in the appendix.

Appendix A.5 shows how to derive the joint log-likelihood of the full observation vectors, $Z_0, \ldots, Z_{t+\Delta t}, \ldots$, where:

$$Z_t = \left( y(t, 3M) \quad \tilde{y}(t, 1Y) \quad \tilde{y}(t, 3Y) \quad \tilde{y}(t, 5Y) \quad \tilde{y}(t, 10Y) \right)$$

$$\log \frac{\Phi_{t-\Delta t}}{\Phi_t} \quad \log \frac{S_{t-\Delta t}}{S_t} \quad \log \frac{Y_{t-\Delta t}}{Y_t} \quad \log \frac{Y_{t-\Delta t}}{Y_t} \right)' \quad (3.4)$$

In practice, the numerical maximisation of the full likelihood raises convergence and stability concerns, due to the high number of parameters to estimate (29) and the non-convexity of the objective function with respect to these parameters. We have therefore followed a different procedure, which consists of maximising a sequence of marginal log-likelihoods:

1. First, we calibrate the “interest rate block” of the model, with a state vector reduced to a single element, $V_t = r_t$, and a vector of observations $Z_t$ containing only the six yields. This gives estimates for $a$, $b$, $\sigma^r$ and for the residuals $s_{1Y}$, $s_{3Y}$, $s_{5Y}$ and $s_{10Y}$;

2. Then we calibrate the “inflation block”, with $V_t = \Phi_t$ and $Z_t = \log \frac{\Phi_{t-\Delta t}}{\Phi_t}$. This gives estimates for $\pi$ and $\sigma^\Phi$;
3. Then we calibrate the correlation $\rho^r$, taking $V_t = (r_t, \log \Phi_t)'$ and $Z_t = (y(t, 3M), \log \frac{S_t}{S_{t-\Delta t}})'$. The parameters for the interest rate and inflation processes are taken equal to the estimates obtained in steps 1 and 2;

4. Then we calibrate the "stock and Sharpe ratio block", with $V_t = (r_t, \log S_t, \lambda^S)'$ and $Z_t = (y(t, 3M), \log \frac{S_t}{S_{t-\Delta t}}, DY_t)'$. We take the parameters of the interest rate process to be equal to the estimates obtained in step 1. This gives estimates for $\sigma^S, \kappa, \lambda, \rho^r$ and $\rho^S$;

5. Then we calibrate the correlations $\rho^\Phi$ and $\rho^\lambda$, with $V_t = (r_t, \log \Phi_t, \log S_t, \lambda^S)'$ and $Z_t = (y(t, 3M), \log \frac{\Phi_t}{\Phi_{t-\Delta t}}, \log \frac{S_t}{S_{t-\Delta t}}, DY_t)'$;

6. Then we calibrate the "real asset block", with $V_t = \log Y_t$ and $Z_t = \log \frac{Y_t}{Y_{t-\Delta t}}$, taking the parameters of the interest rate process equal to the values calibrated in step 1. This gives estimates for $\sigma^Y, \lambda^Y$ and $\rho^Y$;

7. Finally we calibrate the correlations that involve the real asset and that have not been estimated at previous steps, namely $\rho^\Phi^Y$, $\rho^{SV}$ and $\rho^{SY}$. We set $V_t = (r_t, \Phi_t, \log S_t, \lambda^S, \log Y_t)'$ and $Z_t = (y(t, 3M), \log \frac{\Phi_t}{\Phi_{t-\Delta t}}, \log \frac{S_t}{S_{t-\Delta t}}, DY_t, \log \frac{Y_t}{Y_{t-\Delta t}})'$. All other parameters (related to interest rate, inflation, stock, Sharpe ratio and real estate) are taken equal to the point estimates obtained in steps 1, 2, 3, 4, 5 and 6.

It is important to note that this sequential procedure is not equivalent to maximizing the full likelihood of observations contained in the vector (3.5). This procedure, on the other hand, has three advantages. First, it partially overcomes numerical problem that inevitably arise when maximizing a non-concave function by reducing the number of parameters to estimate simultaneously at each step. Second, it enables to use the longest available time series since the sampling period used in each step is not restricted to be the maximal period over which all variables are observed. For example, the time series for the real estate process is much shorter than the other series. If we were to maximize full likelihood, we would ignore many observations of the yields and of stock returns. Third, the procedure avoids introducing spurious dependencies between parameter estimates that belong to different blocks. For instance, we do not want our estimates for the interest rate parameters to depend on the particular time series that is chosen for the stock index in step 3 or for the real estate index in step 4. Proceeding sequentially, we can ensure that if another real estate index is used in step 4, this change will have no feedback effect on the estimates for the interest rate parameters. This technique is close to that employed by Hoevenaars et al. (2008), who estimate a VAR model subject to a series of restrictions. In particular, they impose that alternative asset classes such as real estate, commodities and hedge funds have no impact on the dynamics of traditional classes such as stocks and bonds. With these restrictions, the estimated coefficients of the dynamics of traditional classes do not depend on the estimated coefficients of the dynamics of alternative classes.

3.3 Results

Table 1 displays the estimates for the various parameters. It reports also the standard errors associated with the Maximum Likelihood estimates. Standard errors are computed as the square roots of the diagonal elements of the observed Fisher's information matrix. This matrix is equal to the negative of the Hessian matrix of the log-likelihood function, evaluated at the ML estimates. Since the ML estimator is asymptotically normally distributed around the true parameter value, 95% confidence bounds for true parameter values are obtained by subtracting two standard deviations from the estimate and adding two standard deviations to the estimate. Unsurprisingly in view of the literature on the estimation of expected returns (see Merton (1980)), all parameters that relate to the level of Sharpe ratios are difficult to calibrate. The Sharpe ratios on real estate and commodities – 34.62% and 27.52% – come with substantial standard errors of 24.40% and
19.67%. These errors imply wide confidence intervals, which leaves us with a large amount of uncertainty over the true values for these parameters. The Sharpe ratio of stocks is estimated at 29.05%, with a standard error of 10.82% only, which is slightly better than for the real assets. On the other hand, the parameters $m$ and $p$ introduced in equation (3.2) are estimated with large errors, since the corresponding standard deviations are of the same order of magnitude as the estimates themselves. As a consequence, the affine function of the dividend yield proposed in equation (3.2) yields a relatively imprecise estimate of the expected return. Another parameter that appears to be estimated with low accuracy is the standard deviation of the Sharpe ratio, $\sigma^2$. The point estimate is 12.39%, while the standard deviation is 12.74%. Hence it is hard to assess the degree of variability in the Sharpe ratio of stocks. Overall the estimated standard errors illustrate that the point estimates for the various parameters should be considered with caution, bearing in mind that true parameter values may be substantially different.

Table 1 is completed by figures 1 and 2. Figures 1 shows the implied short-term rate time-series. This series is obtained by solving equation (2.2) for $r_t$, taking the maturity of the perfectly observed yield equal to 3 months, and taking all continuous-time parameters equal to the point Maximum Likelihood estimates. Figure 2 displays the implied Sharpe ratio for the equity index. The series of the implied Sharpe ratio is computed through equation (3.3). The Sharpe ratio takes values over a wide range, with a minimum of $-0.15\%$ on Q4.1999 and a maximum of $94.57\%$ on Q1.1982. Although it seems undesirable to have negative expected excess returns on stocks, the value $-0.15\%$ is not a practical concern because it is only slightly negative, and also because it is the only negative number in the time series. Figure 2 also displays the term structure of risk of excess stock returns, which is formally defined as:

$$\sqrt{\frac{1}{n\Delta t} \sum_{t=1}^{n} \left( \log \frac{S_{t+i}}{S_{t+i-1}} - (\Delta t) y(t_{t+i-1}, \Delta t) \right)},$$

where the horizon $n\Delta t$ takes values 1 quarter, 2 quarters, and so on. Details on the derivation of this term structure are given in appendix A.6. The annualized volatility appears to be decreasing with the horizon. This property is a well-known consequence of the predictability in stock returns: the risk premium on stocks in our model is mean-reverting, and its innovations are negatively correlated with realized returns, which implies that stocks are less risky on the long run. Overall, our model predicts slightly larger long-term annualized volatilities than the VAR models used in the literature. For example, Amenc et al. (2009) report a volatility of 8% per annum when stocks are held for 25 years, which is smaller than the 11.90% predicted by our model. A possible explanation is that the literature incorporates in general several predictors of stock returns, such as the dividend yield, the term spread, the credit spread and the ex-post real return on T-bills among others. Our model takes a more parsimonious approach, in which equity risk premium depends on one mean-reverting state variable only, and this state variable is identified to the dividend yield. It is likely that this simplification leads to underestimate the degree of predictability in stock returns in comparison to more comprehensive models. 11

3.4 Estimating the Price of Inflation Risk

The price of inflation risk, $\lambda(y)$, is not calibrated by likelihood maximization, because we do not expect the theoretical model of section 2 to have sufficient flexibility to fit both the nominal and the real term structures. Instead of using this method, we estimate $\lambda(y)$ directly from nominal and real yields. The breakeven inflation rate is defined as the difference between the nominal and real yields of equal maturities:

$$\text{BEI}(t, \tau) = y(t, \tau) - y^f(t, \tau),$$
where \( y(t, \tau) \) is the nominal yield of maturity \( \tau \) observed at date \( t \), and \( y'(t, \tau) \) is the real yield, defined as:

\[
y'(t, \tau) = -\frac{1}{\tau} \log \frac{I(t, t+\tau)}{\Phi_t}.
\]  

(3.6)

Data for Canadian real yields can be found on Bloomberg at the quarterly frequency from Q2.2000 to Q1.2011.

It follows from the expressions for the prices of nominal and real zero-coupon bonds that:

\[
\text{BEI}(t, \tau) = \pi - \sigma^\Phi \lambda^\Phi - \frac{\sigma^\tau \sigma^\Phi \rho^\Phi}{a} \left[ 1 - \frac{1-e^{-\alpha \tau}}{\alpha \tau} \right].
\]

Since the third term in the right side is of second order importance compared to the first two ones, we have that:

\[
\text{BEI}(t, \tau) - \pi \approx -\sigma^\Phi \lambda^\Phi.
\]

Averaging values of the left side across dates \( t_1, ..., t_n \), we finally obtain that:

\[
\lambda^\Phi \approx -\frac{1}{\sigma^\Phi} \sum_{i=1}^{n} [\text{BEI}(t_i, \tau) - \pi].
\]  

(3.7)

This relationship can be exploited to obtain \( \lambda^\Phi \) using an estimated value for \( \sigma^\Phi \), and the average value of \( B(\cdot, \tau) - \pi \) over the period Q2.2000-Q1.2011. Of course, the estimated \( \lambda^\Phi \) depends on the maturity used for the yields.

In order to have a more realistic view on inflation expectations, we use a series of Canadian expected inflation rate available from Datastream, even though our model assumes the parameter to be constant, with a calibrated value of 3.88% from table 1.\(^{13}\) Equation (3.7) is therefore modified into:

\[
\lambda^\Phi \approx -\frac{1}{\sigma^\Phi} \sum_{i=1}^{n} [\text{BEI}(t_i, \tau) - \pi_{ti}].
\]

where \( \pi_{ti} \) denotes the observed expected inflation rate at date \( t_i \). We obtain estimates of \( \lambda^\Phi \) of 7.86% for \( \tau = 10 \) years, -9.08% for \( \tau = 20 \) years, and -6.31% for \( \tau = 30 \) years. Since we obtain positive as well as negative values, we finally retain a "neutral" value of 0 for the parameter \( \lambda^\Phi.\(^{14}\)

3.5 Choice of Risk Aversion Parameter

Optimal portfolio strategies depend on the relative risk aversion \( \gamma \). Since this parameter is not observable, we set it indirectly by computing the allocation to the PSP at date 0. From proposition 1, the weight allocated to the PSP at date 0 when only stocks and nominal bonds are used is:

\[
\frac{\lambda^\Phi_0^{\text{PSP}}}{\sigma^{PSP}_0} = 1' \left[ (\sigma^\tau)' \sigma^\tau \right]^{-1} (\sigma^\tau)' \lambda_0^1,
\]  

(3.8)

where \( \sigma^1 \) is the volatility matrix of stocks and nominal bonds and \( \lambda^1 \) is the corresponding price of risk vector:

\[
\sigma^1 = \begin{pmatrix} \sigma^B & \sigma^S \end{pmatrix}, \quad \lambda_0^1 = \sigma^1 \left[ (\sigma^1)' \sigma^1 \right]^{-1} \left( -D(\tau_B) \sigma^\tau \lambda^r \right).
\]  

(3.9)

We observe that the ratio (3.8) is independent both from investment horizon and from the maturity of liabilities, since the PSP is the same for all investment horizons and is not impacted by the presence or the absence of liabilities. Taking \( \lambda_0^S \) equal to the long-term mean of the Sharpe ratio, \( \bar{\lambda} \), and the parameter values of table 1, we obtain a value 7.65 for the ratio (3.8). A risk aversion \( \gamma = 10 \) will thus lead to an initial allocation of 76.50% to the PSP, while values \( \gamma = 25 \) and \( \gamma = 50 \) lead respectively to initial allocations of 30.58% and 15.29%. Hence we retain the values 10, 25.

\( ^{13} \) This value will never be used in the numerical computations of this paper, because it does not enter into the expressions for the correlations and expected utilities computed in sections 4 and 5.

\( ^{14} \) The sign of inflation risk premium is controversial in the literature. Evans (1998) and Campbell et al. (2009) argue that it can be negative, while other papers report positive values [see for example Campbell and Viccica (2001) and Buraschi and Jiltsov (2005)]. A zero inflation risk premium is also assumed in Campbell et al. (2003) and in Kothari and Shanken (2004).
and 50 for the risk aversion in the subsequent illustrations. A value of 50 tends to be higher than upper bounds typically considered in the literature for relative risk aversion coefficients, but this is consistent with our focus on liability risk hedging motives.

Obviously, less risk-averse investors would not pay as much attention as more risk-averse to liability and inflation risk hedging, and the concern over hedging these risks would eventually vanish as $\gamma$ converges to 1.

### 4. Is Inflation Risk Prominent in Liability Risk?

As can be seen from equation (2.10), liability risk can be decomposed into two parts, interest rate risk and inflation risk. Interest rate risk can be hedged away by trading in nominal bonds, which are natural ingredients in investors’ asset mix. In contrast, inflation risk is not spanned by stocks, bonds and real assets, and thus can only be completely hedged if inflation-linked bonds are introduced. The purpose of this section is to analyze the importance of unhedgeable risk in total liability risk.

#### 4.1 Short-Term Liability Risk

A simple calculation suggests that if the time-to-maturity of liabilities, $\tau_L - t$, is large, then interest rate risk is dominant in liability risk, so that hedgeable risk dominates unhedgeable risk. We illustrate this fact by computing the absolute value of the instantaneous correlation between LHP and liabilities. This absolute correlation is formally defined as:

$$ \left| \text{corr}_t \left[ \frac{dA_t}{A_t}, \frac{dL_t}{L_t} \right] \right| = \frac{\text{Cov}_t \left[ \frac{dA_t}{A_t}, \frac{dL_t}{L_t} \right]}{\sqrt{\text{Var}_t \left[ \frac{dA_t}{A_t} \right]} \sqrt{\text{Var}_t \left[ \frac{dL_t}{L_t} \right]}}. $$

(4.1)

where $w_t = w_t^{\text{LHP}}$. It follows from the expressions given in proposition 1 that (see details in appendix A.4):

$$ \left| \text{corr}_t \left[ \frac{dA_t}{A_t}, \frac{dL_t}{L_t} \right] \right| = \frac{\| \sigma(\tau - t) \|^{-1} \sigma \sigma^L(\tau_L - t) \|}{\| \sigma^L(\tau_L - t) \|}. $$

(4.1.1)

The numerator is the norm of the orthogonal projection of the volatility vector of the liability portfolio, $\sigma(\tau - t)$, onto the columns of the volatility matrix $\sigma$; and the denominator is the norm of $\sigma^L(\tau_L - t)$. As a consequence, the square of the ratio (4.1) can be interpreted as a percentage of total variance of liabilities that is explained by the traded assets. It is also shown in appendix A.4 that the quantity (4.1) is the maximum correlation between wealth and liability portfolio, and that it is achieved with the following strategy:

$$ w_t = \beta_t^L w_t^{\text{LHP}} = (\sigma^L)^{-1} \sigma \sigma^L(\tau_L - t). $$

(4.2)

In table 2 we consider various investment universes, which lead to different LHPs, and different values for the time to maturity $\tau_L - t$. The maximum value, equal to 15 years, has been chosen to reflect the maximum duration of the liabilities for typical pension plans. It appears that for $\tau_L - t = 15$ years, a portfolio invested in nominal bonds only achieves an absolute correlation of 98.17% with inflation-linked liabilities, which shows that interest rate risk is indeed dominant for such long horizons. This correlation is still at a high level, namely 94.56%, for $\tau_L - t = 5$ years, and eventually decreases to 53.86% in the case of a one-year maturity. Indeed, when the horizon becomes shorter, the component $D(\tau_L - t)\sigma$ in (2.10) shrinks and exposure to un hedgeable inflation risk becomes more important in relative terms. Portfolios formed with either stocks only or one real asset class only do not exhibit a monotonic pattern of correlation as the time-to-maturity changes because their volatility vector is not affected by the remaining horizon. A monotonic pattern is obtained only for LHPs that contain nominal bonds. When at least one asset
class is added to nominal bonds in the LHP, the correlation increases, which is a consequence of a mathematical property of projections: the right side of (4.1) if non-redundant columns are added to the matrix $\sigma$. This increase, however, is marginal for long horizons, since the correlation for a nominal bond only portfolio was already extremely high, as noted above. For portfolios based on nominal bonds, stocks and one real asset class, either commodities or real estate, it is possible to achieve a correlation greater than 98% if the time-to-maturity is 15 years. However, if the time-to-maturity is only 1 year, the correlation reaches to 63.57% at most. Of course, for any time-to-maturity, it is only if inflation-indexed bonds are introduced together with nominal bonds that a perfect correlation can be achieved. Note that if the maturity of the traded indexed bond differs from the time-to-maturity of liabilities, nominal bonds are still needed in order to attain a perfect correlation, as an additional degree of freedom is needed to match the exposure to interest rate risk of the liability portfolio and the LHP.

4.2 Term Structure of Liability Risk

In fact, investors endowed with long-term objectives may not find the correlations in table 2 to be the relevant indicators. To take a long-term perspective, we need to compute the long-term correlations between asset classes and liabilities, and between the LHP and liabilities. This is the focus of this subsection.

The correlations at horizon $T$ between asset classes and liabilities can be obtained using the following expressions (see appendix A.7 for details):

$$
\log \frac{L_T}{L_0} = F^L(T) + \int_0^T \left[ \sigma^L(\tau_L - t) - \sigma^B(T - t) \right]' dz_t, \quad (4.3)
$$

$$
\log \frac{S_T}{S_0} = F^S(T) + \int_0^T \left[ \sigma^S - \sigma^B(T - t) + \sigma^S \frac{1 - e^{-\kappa(T-t)}}{\kappa} \sigma^\lambda \right]' dz_t, \quad (4.4)
$$

$$
\log \frac{Y_T}{Y_0} = F^Y(T) + \int_0^T \left[ \sigma^Y - \sigma^B(T - t) \right]' dz_t, \quad (4.5)
$$

$$
\log \frac{B_T}{B_0} = F^B(T) + \int_0^T \left[ \sigma^B(\tau_B) - \sigma^B(T - t) \right]' dz_t, \quad (4.6)
$$

$$
\log \frac{I_T}{I_0} = F^I(T) + \int_0^T \left[ \sigma^I(\tau_I) - \sigma^B(T - t) \right]' dz_t. \quad (4.7)
$$

The quantities $F(t)$, for $i = L, S, Y, B, I$, are deterministic, so variances and covariances are determined only by the stochastic integrals above. To write the variance of a stochastic integral and the covariance between two stochastic integrals, we can use "Ito isometry" (see Karatzas and Shreve (1991)). For instance, the covariance between liabilities and the stock index can be expressed as:

$$
\text{Cov} \left[ \log \frac{S_T}{S_0}, \log \frac{L_T}{L_0} \right] = \int_0^T \left[ \sigma^L(\tau_L - t) - \sigma^B(T - t) \right]' \left[ \sigma^S - \sigma^B(T - t) + \sigma^S \frac{1 - e^{-\kappa(T-t)}}{\kappa} \sigma^\lambda \right] dz_t,
$$

while the variances of stock return and liabilities are given by:

$$
\text{Var} \left[ \log \frac{L_T}{L_0} \right] = \int_0^T \left\| \sigma^L(\tau_L - t) - \sigma^B(T - t) \right\|^2 dt,
$$

$$
\text{Var} \left[ \log \frac{S_T}{S_0} \right] = \int_0^T \left\| \sigma^S - \sigma^B(T - t) + \sigma^S \frac{1 - e^{-\kappa(T-t)}}{\kappa} \sigma^\lambda \right\|^2 dt.
$$
Figure 3 displays the term structure of correlations of the various asset classes with liabilities, at different horizons. Of course, at the shortest investment horizon considered in the figures (one quarter), the absolute value of the correlation is close to the instantaneous correlation reported in table 2, towards which they would converge in the limit of vanishing maturity. At longer horizons, however, correlations may become very different from these short-term values. For instance, stocks and real assets have higher correlations with the liabilities on the long run than on the short run. For an initial maturity of liabilities equal to 10 or 15 years, these assets are negatively and rather weakly correlated with liabilities at short investment horizons. This means that a strategy entirely invested in stocks or in one of the real asset classes will poorly replicate liabilities on the short run. When the investment horizon is equal to the initial maturity of liabilities, that is when assets are held until liabilities are paid, the correlation becomes positive for stocks, real estate and commodities.

The situation is completely different for bonds, whether nominal or indexed. They have good liability-hedging properties at short horizons, as could be expected from the results in table 2, but their correlation with liabilities is decreasing in the investment horizon, both in absolute and in algebraic value. This means that they are less attractive for liability-hedging purposes when the investment horizon approaches the initial maturity of liabilities. That the correlation between nominal bonds and liabilities is close to 1 at short horizons can be explained by the fact that interest rate risk is dominant in liability risk at such horizons. However, when one approaches the payment date $\tau_L$, inflation risk becomes relatively more important, and since it is only very imperfectly hedged by nominal bonds, the correlation decreases. Inflation-indexed bonds appear to have higher correlations with liabilities at all horizons, but they cannot achieve a perfect correlation with liabilities if they are employed alone. Indeed, they are assumed to have a constant maturity $\tau_I = 10$ years, while liabilities have a vanishing time-to-maturity. It would be only if an inflation-indexed bond with fixed maturity equal to the initial maturity of liabilities were present in the asset mix that a perfect correlation at all investment horizons would be obtained. In the general case, trading in both the nominal bond and the inflation-linked bond is needed to match simultaneously the inflation and interest rate risk exposures in the liability portfolio value.

For comparison purposes, figure 4 shows the term structure of correlations of the various asset classes with inflation, at different horizons. These correlations are computed from equations (4.6) to (4.7) and from the following equality, that follows directly from (2.7):

$$\text{Corr} \left[ \log \frac{A_t}{A_0} \log \frac{L_{\tau L}}{L_0} \right] = \frac{\| \sigma (\sigma^\prime \sigma)^{-1} \sigma^\prime \sigma^* \|}{\| \sigma^* \|},$$

They are distinct from the correlations reported in figure 3, because we have $L_t \neq \Phi_t$ at dates $t$ that precede the liability payment date $\tau_L$. At maturity, on the other hand, we have $L_{\tau L} = \Phi_{\tau L}$, in which case the correlations in figures 3 and 4 are equal. The comparison of both figures highlights the fact that liability-hedging properties may be very different from inflation-hedging properties at horizons shorter than the maturity of liabilities. For instance, nominal bonds are negatively correlated with inflation at all horizons considered here, i.e. at horizons lower than 15 years, while they have strongly positive correlation with liabilities on the short run! Such a contrast is less pronounced but still present for inflation-indexed bonds. Stocks, real estate and commodities also display different correlation patterns with inflation and liabilities. They are positively correlated with inflation at all horizons, while they have a negative correlation with liabilities at short horizons when the initial maturity of liabilities is 10 or 15 years. Overall, figures 3 and 4 show that when there is a significant fraction of interest rate risk in liability risk, which is the case for long times-to-maturity, liability hedging is not just a matter of inflation hedging: it is at least equally important to hedge interest rate risk, that is, to match the duration of liabilities.

15 - For equities, commodities and real estate, the long-term values for correlations with the liabilities (i.e., the value of the correlation at horizon) is approximately the same for all horizons. This is consistent with the fact that the term structure of correlation of the return on these asset classes with inflation is found to be quasi-flat (see figure 4) since liability risk only contains inflation risk at horizon. Using vector-error correcting models would lead to upward slopping term structure (see e.g., Amenc et al. (2009)) but the lack of statistical robustness of such models makes them ill-suited for use in a formal portfolio optimization exercise.

16 - A negative correlation is not a problem per se for liability-hedging purposes since it can be turned into a positive correlation by shorting the asset.
As in the short-term analysis, one may wonder if it is possible to achieve substantially higher correlations with the liability portfolio by mixing the various asset classes. Since the LHP maximises the squared instantaneous correlation with liabilities, it is also a reasonably good candidate for a portfolio achieving a high correlation over the long run. More formally, we consider strategy (4.2), and we derive the correlation between the log-return on wealth, log\((A_T/A_0)\), and the log-return on the liability portfolio, log\((L_T/L_0)\), at various horizons \(\tau_n\). These correlations are derived in analytical form in appendix A.7, and figure 5 shows the term structures. A first observation is that the strategy has positive correlation with liabilities at all horizons. Although not surprising, this property was not completely straightforward, because strategy (4.2) is not designed to maximise the long-term correlation between the log-return on wealth and that on the liability portfolio, but only to maximise the instantaneous correlation. A second noticeable fact is that the long-term correlation increases in the number of assets included in the LHP. Again, this property is common to short-term correlations (in table 2) and long-term ones: an LHP based on nominal bonds and one real asset class achieves higher correlation than an LHP constructed from bonds only. But all tested LHPs have much better liability-hedging qualities at short horizons. The only exception is trivial: it corresponds to a LHP invested in nominal and indexed bonds, that yields a correlation of 100% at any horizon. In contrast, imperfect LHPs have lower correlations with liabilities at horizons close to maturity. Of particular interest is the correlation obtained when the LHP is held until the payment date of liabilities. Appendix A.7 shows that this correlation is independent from \(\tau\), and is given by:

\[
\text{Corr} \left[ \log \left( \frac{A_{\tau L}}{A_0} \right), \log \left( \frac{L_{\tau L}}{L_0} \right) \right] = \frac{\| (\sigma' \sigma)^{-1} \sigma' \Phi \|}{\| \sigma' \Phi \|},
\]

where \(\sigma\) is the volatility matrix of assets included in the LHP.

This ratio is reminiscent of the short-term correlation (4.1), but the volatility vector of the liability portfolio is replaced by that of the price index. In particular, if only nominal bonds are included in the LHP, the final correlation is equal to \(\rho^{n\Phi}\), that is 16.49%. For a portfolio of nominal bonds and real estate, the final correlation is 28.97%, and it is equal to 43.09% if commodities are used together with nominal bonds. Overall, these correlations at horizon are far from 100%, which suggests that there remains a substantial part of unhedgeable liability risk.

4.3 Utility Cost of Imperfect Inflation Hedging

The correlation between wealth and liabilities provides a measure of the fraction of long-term liability risk that is hedged by traded assets, but it does not fully represent investor’s preferences. Instead, the utility of the terminal funding ratio \(A_T/L_T\) measures the welfare generated by a given allocation strategy. To assess the welfare loss induced by the impossibility to hedge completely liability risk, we compute the Monetary Utility Loss (MUL) of a strategy that employs only nominal bonds or only nominal bonds and stocks in the LHP, with respect to a strategy that also uses inflation-indexed bonds.

In order to isolate the benefits of inflation-indexed bonds from a liability-hedging perspective, we keep the composition of the PSP and of the Sharpe ratio-hedging portfolio constant. These two portfolios are assumed to be invested in stocks and bonds only. In particular, they do not include indexed bonds, since the primary objective of these bonds is to hedge inflation risk. Formally, we let \(\sigma^1\) be the volatility matrix of stocks and nominal bonds only, \(\lambda^1\) be the corresponding price of risk vector, \(\sigma^2\) be the volatility matrix of stocks, nominal bonds and indexed bonds, and \(\lambda^2\) be the associated price of risk vector:

\[
\sigma^2 = \begin{pmatrix} \sigma^B & \sigma^S & \sigma^I \end{pmatrix}, \quad \lambda^2 = \sigma^2 \left[ (\sigma^2)' \sigma^2 \right]^{-1} \begin{pmatrix} -D(\tau_B) \sigma^r \lambda^r \\ \sigma^S \lambda^S \\ -D(\tau_I) \sigma^r \lambda^r + \sigma^\Phi \lambda^\Phi \end{pmatrix}.
\]
We remind that $\sigma^i$ and $\lambda^i$ are defined in equation (3.9). In universes 1 and 2, we define the following two portfolios:

$$W_{\text{PSP,1}}^t = \frac{1}{1' \left[ (\sigma^1)' \sigma^1 \right]^{-1} (\sigma^1)' \lambda^1_t} \left( \begin{array}{c} (\sigma^1)' \sigma^1 \end{array} \right)^{-1} \left( (\sigma^1)' \lambda^1_t \right) \begin{array}{c} 0 \end{array}, \quad (4.9)$$

$$W_{\lambda,1}^t = \frac{1}{1' \left[ (\sigma^1)' \sigma^1 \right]^{-1} (\sigma^1)' \sigma^\lambda} \left( \begin{array}{c} (\sigma^1)' \sigma^1 \right)^{-1} \left( (\sigma^1)' \sigma^\lambda \right) \begin{array}{c} 0 \end{array} \right), \quad (4.10)$$

$W_{\text{PSP,1}}$ is therefore a portfolio that allocates the same relative weights to stocks and bonds as the PSP computed over stocks and bonds only, and assigns a zero weight to indexed bonds. The portfolio $W_{\lambda,1}^t$ has a similar interpretation. We denote with $\lambda^{\text{PSP,1}}_t$ and $\sigma^{\text{PSP,1}}_t$ the Sharpe ratio and the volatility of the portfolio $W_{\text{PSP,1}}^t$, and with $\beta^{\lambda,1}_t$ the beta of the Sharpe ratio with respect to the portfolio $W_{\lambda,1}^t$. Both the strategies we test will make use of the portfolios $W_{\text{PSP,1}}$ and $W_{\text{PSP,2}}$. We then consider one version of the LHP for each universe:

$$W_{t}^{\text{LHP,1}} = \frac{1}{1' \left[ (\sigma^1)' \sigma^1 \right]^{-1} (\sigma^1)' \sigma^{L(t)}} \left( \begin{array}{c} (\sigma^1)' \sigma^1 \right)^{-1} \left( (\sigma^1)' \sigma^{L(t)} \right) \begin{array}{c} 0 \end{array}, \quad (4.11)$$

$$W_{t}^{\text{LHP,2}} = \frac{1}{1' \left[ (\sigma^2)' \sigma^2 \right]^{-1} (\sigma^2)' \sigma^{L(t)}} \left( \begin{array}{c} (\sigma^2)' \sigma^2 \right)^{-1} \left( (\sigma^2)' \sigma^{L(t)} \right) \begin{array}{c} 0 \end{array} \right) = \frac{1}{1 + \frac{D(tL - t) - D(t_1)}{D(t_B)}} \left[ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right] + \frac{D(tL - t) - D(t_1)}{D(t_B)} \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right].$$

The former LHP is the portfolio of stocks and bonds that maximizes the squared correlation with the innovations to the liability portfolio, while the latter LHP is computed over stocks, bonds and inflation-indexed bonds. Only the second LHP allows for a perfect matching of liabilities, as is clear from the previous results (see table 2 and figure 5). Note that in general, the second LHP is not invested in the indexed bond only: due to the difference between the duration of the traded indexed bond and that of liabilities, a duration adjustment is needed, which requires the use of nominal bonds. The respective betas of the liability portfolio with respect to the two LHPs are given by:

$$\beta^{L,1}_t = 1' \left[ (\sigma^1)' \sigma^1 \right]^{-1} (\sigma^1)' \sigma^{L(t)} \begin{array}{c} 0 \end{array},$$

$$\beta^{L,2}_t = 1 + \frac{D(tL - t) - D(t_1)}{D(t_B)}. \quad (4.12)$$

Because the traded inflation-indexed bond is assumed to have a constant maturity and liabilities have a vanishing time-to-maturity, the coefficient $\beta^{L,2}_t$ cannot be equal to 1 at all dates, and $W_{t}^{\text{LHP,2}}$ is not fully invested in inflation-indexed bonds at all dates.

With this notation, the two portfolio rules that we compare are given by:

$$w_i^t = \lambda^t_{\text{PSP,1}} W_{t}^{\text{PSP,1}} + \left( 1 - \frac{1}{\gamma} \right) \beta^{L,i}_t W_{t}^{\text{LHP,i}} - \left( 1 - \frac{1}{\gamma} \right) \left[ C_2(t - t) + C_3(t - t) \lambda^S_2 \right] W_{t}^{\lambda,i},$$

for $i = 1, 2$. The functions $C_3$ and $C_2$ in this equation are given by equations (A.3) and (A.4) in appendix A.2, with:

$$\Lambda^1 = \sigma^2 \left[ (\sigma^2)' \sigma^2 \right]^{-1} \begin{array}{c} -D(t_B) \sigma^\gamma \lambda^r \\ 0 \end{array}, \quad \Lambda_2^2 = \sigma^2 \left[ (\sigma^2)' \sigma^2 \right]^{-1} \begin{array}{c} 0 \\ \sigma^\phi \lambda^\phi \end{array},$$

$$N = I_3 - \sigma^2 \left[ (\sigma^2)' \sigma^2 \right]^{-1} \begin{array}{c} (\sigma^2)' \sigma^2 \end{array}. \quad (4.17)$$

The reason why we include the volatility vector of inflation and the price of inflation risk in the definition of $\Lambda^1$, $\Lambda^2$ and $N$ is that these quantities refer to the optimal portfolio strategy in the market with nominal bonds, stocks and indexed bonds. We could also have chosen to compute the functions $C_i$ and $C$ related to the optimal strategy in the market with nominal bonds and stocks only.
The MUL of strategy $\omega^1$ with respect to strategy $\omega^2$ is computed as explained in section 2.3. It is the fraction by which the capital invested at time 0 in strategy $\omega^1$ must be increased so as to yield the same expected utility as strategy $\omega^2$. Since $\omega^1$ and $\omega^2$ only differ through the composition of the LHP – which includes nominal bonds and stocks only for $\omega^1$, and nominal bonds, stocks and indexed bonds for $\omega^2$ –, the MUL can be taken as a measure of the utility loss incurred by giving up indexed bonds in the LHP. The solid lines on figure 6 show the MULs for different levels of risk aversion and different investment horizons. The MUL appears to be increasing in the risk aversion parameter, which happens for two reasons. First, with a higher $\gamma$, the investor allocates more to the LHP. Any difference in the composition of the LHP will thus be magnified. Second, a higher $\gamma$ means that the investor penalizes more the uncertainty in his final funding ratio. The MUL also appears to be increasing in the investment horizon. This effect is straightforward: a longer horizon magnifies the effects of imperfect liability hedging.

In practice, it is often the case that pension funds do not use stocks in their LHP. It is thus interesting to measure the utility loss that follows from using only nominal bonds in the LHP. Formally, we consider a third version of the LHP, invested in nominal bonds only:

$$W^{LHP,3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$  

The beta of the liability portfolio with respect to this new LHP is:

$$\beta_t^{L,3} = \frac{(\sigma^B)^t \sigma^L (\tau_L - t)}{\|\sigma^B\|^2} = \frac{D(\tau_L - t)}{D(\tau_B)} - \frac{\sigma^\phi \rho^\phi}{\sigma^\gamma D(\tau_B)}.$$  

With this notation, we consider the strategy $\omega^3$ defined by:

$$w^3_t = \frac{\lambda^{PSP,1}_t}{\gamma \sigma^{PSP,1}_t} W^{PSP,1}_t + \left(1 - \frac{1}{\gamma}\right) W^{LHP,3}_t - \left(1 - \frac{1}{\gamma}\right) \left[C_2(T - t) + C_3(T - t) \lambda^*_t\right] W^{\lambda,1}_t.$$  

The dotted lines on figure 6 represent the MUL of strategy $\omega^3$ with respect to strategy $\omega^2$. Unsurprisingly, the MUL increases if stocks are removed, because a LHP made of nominal bonds only does not track liabilities as accurately as a LHP of nominal bonds and stocks. But the increase in MUL is very small, which shows that the marginal utility gain of adding stocks in a LHP of nominal bonds is minor.

Overall, we confirm that the MULs reported in figure 6 do not seem prohibitively high over short-horizons, but more substantial welfare gains/losses are obtained as the time-horizon converges towards the liability maturity date. For example, if $\gamma = 50$ (which corresponds to a conservative initial allocation of 15.29% in the PSP), the MULs are less than 1% of initial wealth for a liability maturity of one year, but they can reach almost 15% at horizon for a maturity of 15 years. To help understand whether such values are economically significant or not, we have computed for comparison purposes the utility loss that follows from ignoring predictability in stock returns. In this exercise, we have assumed that the investor has access only to nominal bonds, stocks and cash, and has the choice between the following two options: (a) follow the strategy that is actually optimal in the presence of stochastic interest rate and stochastic equity risk premium, as described in proposition 1; (b) ignore the predictability in excess stock returns and treat $\lambda^S$ as if it was a constant parameter equal to $\bar{\lambda}$. In the latter case, of course, the hedging demand against equity risk premium becomes zero and the PSP is a fixed-mix strategy. Figure 7 shows the MULs for different investment horizons and different levels of risk aversion. Of course, the utility cost of treating the Sharpe ratio as a constant is virtually zero for short horizons, i.e., horizons below 2 years, because predictability does not play any role at such low horizons. Moreover, this cost is
decreasing in the risk aversion because ignoring predictability hurts less investors who have only a small fraction of their holdings invested in stocks. This property is different from the one that was obtained when measuring the utility cost of imperfect liability hedging: indeed, the MUL of giving up indexed bonds in the LHP has been shown to be increasing in the risk aversion. Comparing the utility losses induced by imperfect liability hedging to those in figure 6, it can be seen that the utility cost of ruling out inflation-indexed bonds from the LHP is larger than that of ignoring predictability for highly risk-averse investors ($\gamma = 50$). But, as we have seen in section 3, a risk aversion of 50 implies that the weight of the PSP is 15.29% only, which is a conservative allocation. For more aggressive investors ($\gamma = 10$), ignoring predictability carries a larger cost than using an imperfect LHP. For the intermediate risk aversion level $\gamma = 25$, the two utility losses are of comparable magnitude.

5. Benefits of Real Assets for Liability Hedging and Diversification Purposes

In this section we assess the benefits of real assets both from a liability-hedging perspective and from a global perspective, that encompasses both their diversification properties and their ability to hedge inflation risk and equity premium risk. More specifically, we consider real estate and commodities.

5.1 Utility Cost of not Including Real Assets in the LHP

In an attempt to measure the welfare loss incurred by excluding real assets from the LHP, we now compare two strategies with imperfect LHPs. The first one uses nominal bonds and stocks only, while the second one allows for either commodities or real estate in the LHP. For the time being, we do not include real assets in the other building blocks (the PSP and the portfolio hedging Sharpe ratio risk). Formally, we let $\sigma^3$ be the volatility matrix of nominal bonds, stocks and commodities, and $\lambda^3$ be the corresponding price of risk vector:

$$\sigma^3 = \begin{pmatrix} \sigma^B & \sigma^S & \sigma^Y \end{pmatrix}, \quad \lambda^3 = \sigma^3 \left( \begin{pmatrix} (\sigma^3)' \sigma^3 \end{pmatrix}^{-1} \begin{pmatrix} -D(\tau_P)\sigma^Y + \lambda^Y \\ \sigma^S \lambda^S_t \\ \sigma^Y \lambda^Y \end{pmatrix} \right).$$

As substitutes for the actual PSP and the actual Sharpe-ratio hedging portfolio, we consider the same portfolios as above, namely $W_{\text{PSP},1}$ and $W_{\lambda,1}$ defined in equations (4.9) and (4.10). The LHP of stocks and nominal bonds that we use for the first strategy is defined in equation (4.11), while the LHP of nominal bonds, stocks and real asset that we use in the second strategy is defined by:

$$W_{t,LHP,3} = \frac{1}{1' (\sigma^3)' (\sigma^3)^{-1} (\sigma^3)' \sigma^3} \left( \begin{pmatrix} (\sigma^3)' \sigma^3 \end{pmatrix}^{-1} (\sigma^3)' \sigma^3 \right)^{-1} (\sigma^3)' \sigma^3 (\tau_L - t).$$

We denote by $L,3$ the beta of the liability portfolio with respect to this LHP. With this notation, the strategy with nominal bonds and stocks only is defined by:

$$w^4_t = \frac{\lambda_{t,\text{PSP},1}}{\gamma \sigma^3_{t,\text{PSP},1}} W_{t,\text{PSP},1} + \left( 1 - \frac{1}{\gamma} \right) \beta_{L,1} W_{t,LHP,1} - \left( 1 - \frac{1}{\gamma} \right) \left[ C_2(T - t) + C_3(T - t) \lambda^S_t \right] W_{t,\lambda,1},$$

while the strategy that allows real asset to enter the composition of the LHP is defined by:

$$w^5_t = \frac{\lambda_{t,\text{PSP},1}}{\gamma \sigma^3_{t,\text{PSP},1}} W_{t,\text{PSP},1} + \left( 1 - \frac{1}{\gamma} \right) \beta_{L,3} W_{t,LHP,3} - \left( 1 - \frac{1}{\gamma} \right) \left[ C_2(T - t) + C_3(T - t) \lambda^S_t \right] W_{t,\lambda,1}.$$
Figures 8 and 9 show the MUL of strategy \( w_4 \) with respect to strategy \( w_5 \) when the real asset is either the commodity index or the real estate index. This MUL measures the utility loss of removing one real asset class from the LHP. The MUL is increasing in the investment horizon, which happens for the same reason as in figure 6: the benefits of better liability hedging are more visible over a long period, and reach 4.5% for an investment horizon and an initial maturity of liabilities equal to 15 years. Overall, the utility costs of removing real assets from the LHP does not appear to be prohibitive over short-horizons, which is consistent with the previous results showing modest welfare losses from excluding inflation-linked bonds, which have superior (perfect) liability-hedging properties from the LHP. As a result, from a short- to medium-term perspective, a LHP composed of nominal bonds, stocks and commodities or real estate does not represent a significant improvement over a LHP made of nominal bonds and stocks only.

5.2 Utility Cost of not Including Real Assets in the Optimal Allocation

Beyond their usefulness (long-term perspective), or lack thereof (short-term perspective), for liability-hedging purposes, real assets (and inflation-linked bonds) may also be useful in the performance-seeking component of investors’ portfolios due to their positive risk premia and their relatively low correlations with stocks and bonds, which makes them attractive for diversification purposes. To investigate more formally the benefits of including real assets in the PSP, we consider the following strategies: the strategy (5.2), which is entirely invested in stocks and nominal bonds, and the strategy:

\[ w_t^6 = \frac{\lambda_{PSP,2}^t}{\gamma_{PSP,2}^t} W_t^{PSP,2} + \left( 1 - \frac{1}{\gamma} \right) \beta_{L,1}^t W_t^{LHP,1} - \left( 1 - \frac{1}{\gamma} \right) \left[ C_2(T - t) + C_3(T - t) \lambda_t^S \right] W_t^{\Lambda,1}, \]

where the LHP and the Sharpe-ratio hedging portfolio are given in equations (4.10) and (4.11), and the PSP also includes commodities:

\[ W_t^{PSP,2} = \frac{1}{1' \left[ (\sigma_3')\sigma_3 \right]^{-1} (\sigma_3')\lambda_3^S} \left[ (\sigma_3')\sigma_3 \right]^{-1} (\sigma_3')\lambda_3^S. \]

The volatility matrix \( \sigma_3 \) and the price of risk vector \( \lambda_3 \) over the universe made of nominal bonds, stocks and commodities are given in equation (5.1). \( \lambda_{PSP,2}^t \) and \( \sigma_{PSP,3}^t \) denote respectively the Sharpe ratio and the volatility of the portfolio \( W^{PSP,2} \), which is the PSP over the universe of nominal bonds, stocks and commodities.

We then compute the MUL of strategy \( \omega^4 \) with respect to strategy \( \omega^5 \). The results are shown in figures 10 and 11. It can be noted that the MUL is not necessarily a monotonic function of risk aversion, in contrast to what was observed in figures 8 and 9. The reason is that real assets are now included in all building blocks, not only in the LHP. Hence the MULs that we compute here aggregate the benefits of real asset in terms of liability hedging, diversification (within the PSP), as well as possibly Sharpe-ratio risk hedging within the dedicated intertemporal hedging demand. The utility losses of removing real assets from all building blocks are obviously larger than those of ignoring them only in the LHP. For investment horizons of 15 years, we even obtain that the utility cost from excluding real estate may reach 34% for an aggressive investor, i.e., and investor who allocates 76.05% of his wealth to the PSP at time 0. These results therefore suggest that real assets provide benefits to long-term investors, even in situations where their usefulness is limited in terms of inflation and liability hedging purposes.
6. Conclusion
This paper proposes an empirical analysis of the opportunity gains (costs) involved in introducing (removing) various financial and real assets with attractive inflation-hedging properties for long-term investors facing inflation-linked liabilities. From the perspective of analysing various asset classes, investors with inflation-linked liabilities face a trade-off when approaching liability risk management in the absence of inflation-linked bonds. On the one hand, from a short-term perspective, nominal bonds appear to be the best liability hedge because of their ability to generate perfect hedge for interest rate risk. On the other hand, they offer very poor liability hedging benefits from a long-term perspective because of their inability to hedge inflation risk. Conversely, real estate and even more commodities have relatively attractive inflation hedging properties that make them relatively well-suited for liability hedging from a long-term perspective, but their weak interest rate hedging properties make them poor liability hedges on the short run. Using formal intertemporal spanning tests, we find that interest rate risk dominates inflation risk so dramatically within instantaneous liability risk that introducing or removing inflation-linked bonds, or real estate and commodities, from their liability-hedging portfolio has very little impact on investors' welfare from a short-term perspective, in spite of the perfect, or at least attractive, inflation-hedging benefits of these assets. More substantial welfare gains/losses are obtained as the time-horizon converges towards the maturity date of the liabilities and as the relative importance of inflation risk within liability risk increases. For reasonable parameter values, these welfare gains/losses, however, are found to be of the same order of magnitude as those related to ignoring the predictability in stock returns. Even more substantial utility gains are obtained if these asset classes are also used in the performance-seeking portfolio, where they provide diversification benefits with respect to equity returns.

Our work could be extended in several dimensions. We could first provide a thorough robustness analysis of the results in the paper, by trying to assess the impact of changes in various key parameter values. Among these, inflation uncertainty and inflation risk premium obviously stand out as the most important parameters. One way to do this would consist in analysing the situation from the standpoint on different regions in the world, where inflation time-series tend to have very different behaviours. We may also try and see how our results might be affected by the introduction of a stochastic expected inflation component, as opposed to assuming a constant expected inflation as we do in this paper. While the intuition suggests that the results we obtain should not be severely impacted since it is precisely unexpected inflation risk that deserves to be managed, it would be useful to confirm this result through a formal analysis. Finally, the analysis in this paper does not tell us what would be the marginal welfare impact of improving inflation/liability hedging properties of asset classes that are substantial ingredients in the performance-seeking portfolio. Indeed, while the separation theorem implies a distinct focus on liability-hedging versus performance-seeking, it does not imply that an improvement in liability-hedging properties of asset classes heavily present in the performance-seeking portfolio would not be an attractive feature. In particular, we leave for further research an analysis of the welfare benefits of equity portfolios constructed to have improved long-term liability hedging (that is, mostly inflation hedging) properties.

A. Appendix
A.1 Dynamics of Constant-Maturity Indexed Bonds
Consider the strategy which consists of rolling over indexed zero-coupon bonds of time-to-maturity \( \tau_i \). The roll-over takes place in discrete time, at dates \( t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \ldots \). Over the period \([ t_i, t_{i+1} ]\), the log return on the strategy is:

\[
\log \frac{I(t_{i+1}, \tau_i + t_i)}{I(t_i, \tau_i + t_i)} = \log \frac{\Phi_{t_{i+1}}}{\Phi_{t_i}} - D(\tau_i - \Delta t)[r_{t_{i+1}} - r_t] + [D(\tau_i) - D(\tau_i - \Delta t)]r_t + E(\tau_i - \Delta t) - E(\tau_i).
\]
We let $\Delta t$ shrink to zero, set $t_i = t$, and ignore terms of order higher than 1. Hence, for an infinitesimal time increment $dt$:

$$
\log \frac{I(t + dt, \tau_l + t)}{I(t, \tau_l + t)} = d \log \Phi_t - D(\tau_l) \, dr_i + \dot{D}(\tau_l) r_i \, dt - \dot{E}(\tau_l) \, dt
$$

$$
= \left[ \pi - \frac{(\sigma^\phi)^2}{2} - D(\tau_l) a(b - r_l) + \dot{D}(\tau_l) r_i - \dot{E}(\tau_l) \right] \, dt
$$

$$
+ \left[ -D(\tau_l) \sigma^r + \sigma^\phi \right]' \, dz_t.
$$

Substituting the expressions for $E$ and $D$, we obtain the continuous-time log return on the strategy:

$$
d \log I_t = \left[ r_t - D(\tau_l) \sigma^r \lambda^r + \sigma^\phi \lambda^\phi - \frac{1}{2} \| -D(\tau_l) \sigma^r + \sigma^\phi \|^2 \right] \, dt + \left[ -D(\tau_l) \sigma^r + \sigma^\phi \right]' \, dz_t.
$$

From Ito’s lemma, the arithmetic return on the strategy is thus:

$$
\frac{dI_t}{I_t} = \left[ r_t - D(\tau_l) \sigma^r \lambda^r + \sigma^\phi \lambda^\phi \right] \, dt + \left[ -D(\tau_l) \sigma^r + \sigma^\phi \right]' \, dz_t.
$$

A.2 Proof of Proposition 1

The first-order optimality condition in (2.13) implies that there exists a Lagrange multiplier $\eta$ such that:

$$
X^* = L_T U_t^{-1} (\eta M_T L_T) = \eta^{-1/\gamma} L_T^{1-1/\gamma} M_T^{-1/\gamma}.
$$

$X^*$ is only a candidate optimal terminal wealth: it may not be replicable, due to market incompleteness. We now define:

$$
X_t^* = \mathbb{E} \left[ \frac{M_T X^*}{M_t} \right] = \eta^{-1/\gamma} G_t M_t^{-1/\gamma} L_t^{1-1/\gamma},
$$

where:

$$
G_t = \mathbb{E} \left[ \left( \frac{M_T L_T}{M_t L_t} \right)^{1-1/\gamma} \right]. \quad (A.1)
$$

The optimal terminal wealth is obtained by taking $M = M^*$, the minimax pricing kernel of He and Pearson (1991). We assume that the associated process $G$ is a function of $(t, \lambda^S_t)$. This can be written as:

$$
G_t = g(t, \lambda^S_t).
$$

Then, applying Ito’s lemma to (A.1), we obtain that:

$$
\frac{dX_t^*}{X_t^*} = \left[ \frac{g_t}{g} + \frac{g_t}{g} \kappa(\lambda - \lambda_t^S) + \frac{g_t}{g} \left( \sigma^\lambda \right)^2 + r_t + \frac{1 + \gamma}{2} \left( \| \lambda_t \|^2 + \| \nu_t \|^2 \right) \right] \, dt
$$

$$
+ \left[ -\frac{1}{2\gamma^2} \| \sigma^L (\tau_L - t) \|^2 + \frac{g_t}{g} \left( \lambda_t + \nu_t \right)' \sigma^\lambda \right] \, dt
$$

$$
+ \left[ \frac{1}{\gamma} \left( \lambda_t + \nu_t \right)' \sigma^L (\tau_L - t) \right] \, dt + \left[ \frac{1}{\gamma} \left( \lambda_t + \nu_t \right)' \sigma^L (\tau_L - t) + \frac{g_t}{g} \right] \, dz_t.
$$

Since the volatility vector of the process $(X_t^*)_t$ is spanned by traded assets, we have:

$$
\nu_t^* = (1 - \gamma) N \sigma^L (\tau_L - t) - \frac{g_t}{g} N \sigma^\lambda.
$$
Moreover, the process \((M_t^s X_t^s)_t\) follows a martingale, so its drift term is zero:

\[
0 = \left(1 - \frac{1}{\gamma}\right) \left[-D(\tau_L - t)\sigma^r \lambda^r + \sigma^p \lambda^p\right] + \frac{1 - \gamma}{2\gamma^2} (\|\lambda_t\|^2 + \|\nu^*_t\|^2 + \|\sigma^L(\tau_L - t)\|^2) \\
- \left(1 - \frac{1}{\gamma}\right)^2 (\lambda_t + \nu^*_t) \lambda^L(\tau_L - t) - \left(1 - \frac{1}{\gamma}\right) \frac{g_t}{g} (\lambda_t + \nu^*_t - \sigma^L(\tau_L - t))' \lambda^L \\
+ \frac{g_t}{g} \gamma (\lambda_t - \lambda_t^S) + \frac{g_t \lambda}{2g} (\sigma^L)^2. \tag{A.2}
\]

Let us conjecture that \(g\) is of the exponential affine form:

\[
g(t, \lambda_t^S) = \exp \left[\frac{1 - \gamma}{\gamma} \left[C_1(T - t) + C_2(T - t)\lambda_t^S + \frac{1}{2} C_3(T - t) (\lambda_t^S)^2\right]\right],
\]

with \(C_1(0) = C_2(0) = C_3(0) = 0\). Substituting the relevant partial derivatives of \(g\) into the right side of (A.2), we obtain a linear combination of \((1, \lambda_t^S, (\lambda_t^S)^2)\). This linear combination is identically zero, so the coefficients of \(1, \lambda_t^S\) and \((\lambda_t^S)^2\) must be zero. Canceling the coefficient of the quadratic term, we obtain:

\[
\hat{C}_3 = \frac{1 - \gamma}{\gamma} \left[ (\sigma^L)^2 - (1 - \gamma) (\sigma^L)' N \sigma^L \right] C_2^2 - 2 \left[ \kappa - \frac{1 - \gamma}{\gamma} (\sigma^L)' A^2 \right] C_3 + \frac{||A^2||^2}{\gamma}. \tag{A.3}
\]

Cancelling the coefficient of the linear term, we obtain:

\[
\hat{C}_2 = (\lambda^L)' \lambda^L \tau_L - T + \cdot \frac{\left(A^L\right)' \left[ A^1 - \sigma^L(\tau_L - T + \cdot) \right]}{\gamma} - \left[ \kappa - \frac{1 - \gamma}{\gamma} (\sigma^L)' A^2 \right] C_2 \\
+ \frac{1 - \gamma}{\gamma} \left[ (\sigma^L)^2 - (1 - \gamma) (\sigma^L)' N \sigma^L \right] C_2 C_3 \\
+ \left[ \kappa \lambda + \frac{1 - \gamma}{\gamma} (\sigma^L)' [A^1 - \sigma^L(\tau_L - T + \cdot)] + \frac{(1 - \gamma)^2}{\gamma} (\sigma^L)' N \sigma^L(\tau_L - T + \cdot) \right] C_3. \tag{A.4}
\]

Finally, canceling the coefficient of the constant term, we get that:

\[
\hat{C}_1 = D(\tau_L - T + \cdot) \sigma^r \lambda^r - \sigma^p \lambda^p + (\lambda^L)' \lambda^L(\tau_L - T + \cdot) + \frac{||A^1 - \sigma^L(\tau_L - T + \cdot)||^2}{2\gamma} \\
- \frac{(1 - \gamma)^2}{2\gamma} \sigma^L(\tau_L - T + \cdot)' N \sigma^L(\tau_L - T + \cdot) + \frac{1}{2} (\sigma^L)^2 C_3 \\
+ \frac{1 - \gamma}{2\gamma} \left[ (\sigma^L)^2 - (1 - \gamma) (\sigma^L)' N \sigma^L \right] C_2^2 \\
+ \left[ \kappa \lambda + \frac{1 - \gamma}{\gamma} (\sigma^L)' [A^1 - \sigma^L(\tau_L - T + \cdot)] + \frac{(1 - \gamma)^2}{\gamma} (\sigma^L)' N \sigma^L(\tau_L - T + \cdot) \right] C_2. \tag{A.5}
\]

A.3 Proof of Proposition 2
We conjecture that the expected utility from terminal real wealth when (2.15) is followed can be decomposed into the product of utility from current real wealth and an exponential quadratic function of the state variables:

\[
E_t \left[ U \left( \frac{A_t}{L_t} \right) \right] = \frac{1}{1 - \gamma} \left( \frac{A_t}{L_t} \right)^{1 - \gamma} h(t, \lambda_t^S)^\gamma, \\
\text{where} \quad h(t, \lambda_t^S) = \exp \left[ \frac{1 - \gamma}{\gamma} \left[ E_1(T - t) + E_2(T - t)\lambda_t^S + \frac{1}{2} E_3(T - t) (\lambda_t^S)^2 \right] \right].
\]
Being a conditional expectation, the process \((A/L)^{1-\gamma} h(\cdot, \lambda^S)^\gamma\) is a martingale, so it has zero drift. This conditions leads to:

\[
0 = (1-\gamma) \left[ H_1(T-t) + \lambda^S h_3(T-t) \right]' \lambda_t + \frac{(1-\gamma)(2-\gamma)}{2} \| \sigma^L(\tau_L - t) \|^2 \\
- \left(1-\gamma\right)^2 \left[ H_1(T-t) + \lambda^S h_3(T-t) \right]' \sigma^L(\tau_L - t) \\
- \frac{\gamma(1-\gamma)}{2} \left( H_1(T-t) + \lambda^S h_3(T-t) \right)' \left( \begin{array}{c} \sigma^L(\tau_L - t) \\ \sigma^L(\tau_L - t) \end{array} \right) \\
+ \gamma \left( h_t h_\lambda + \frac{h_{\lambda\lambda}}{h} (\lambda - \lambda^S)^2 \right) + \frac{1}{2} (\sigma^L)^2 h_{\lambda\lambda}/h^2 \\
+ \gamma(1-\gamma) \left[ H_1(T-t) + \lambda^S h_3(T-t) - \sigma^L(\tau_L - t) \right]' \sigma^L(\tau_L - t). \tag{A.16}
\]

The relevant partial derivatives of \(h\) are given by:

\[
\frac{h_\lambda}{h} = \frac{1-\gamma}{\gamma} \left[ E_2(T-t) + E_3(T-t) \lambda_t^S \right], \\
\frac{h_{\lambda\lambda}}{h} = \frac{1-\gamma}{\gamma} \left[ E_3(T-t) + \frac{1-\gamma}{\gamma} \left[ E_2(T-t) + E_3(T-t) \lambda_t^S \right]^2 \right].
\]

Substituting these expressions back into the right side of (A.6), we obtain a polynomial function of \(\lambda_t^S\), that must be equal to zero for all values of \(\lambda_t^S\). Hence the coefficients of the polynomial function must be equal to zero. Writing that the coefficient of the squared term is zero, we obtain that:

\[
\hat{E}_3 = (1-\gamma) \left( \sigma^L \right)^2 E_3^2 + 2 \left[ (1-\gamma) \left( \sigma^L \right)' H_2 - \kappa \right] E_3 + 2 H_2^2 \lambda^2 - \gamma H_2^2.
\]

Writing that the coefficient of the linear term is zero, we obtain that:

\[
\hat{E}_2 = H_1' \lambda^2 + H_2' \left[ \lambda^1 + (1-\gamma) (\sigma^L(T_L - T + \cdot)) \right] - \gamma H_1' H_2 + (1-\gamma) \left( \sigma^L \right)^2 E_3 \\
+ \left[ \kappa \lambda + (1-\gamma) \left( H_1 - \sigma^L(T_L - T + \cdot) \right)' \sigma^L \right] E_3 + \left[ -\kappa + (1-\gamma) H_2' \sigma^L \right] E_2,
\]

and writing that the constant term is zero, we obtain that:

\[
\hat{E}_1 = D(\tau_L - T + \cdot) \sigma^L - \sigma^B \lambda^P + H_1' \lambda^1 - (1-\gamma) H_1' \sigma^L(\tau_L - T + \cdot) - \frac{\gamma}{2} H_1'^2 \\
+ \frac{2-\gamma}{2} \left( \sigma^L(T_L - T + \cdot) \right)^2 + \frac{1-\gamma}{2} \left( \sigma^L \right)^2 E_3 \\
+ \left[ \kappa \lambda + (1-\gamma) \left( \sigma^L \right)' \left( H_1 - \sigma^L(T_L - T + \cdot) \right) \right] E_2.
\]

A.4 Properties of Strategy (4.2)

We omit the argument of the vector \(\sigma^L\) for notational simplicity. The correlation between \(\text{LHP}\) and liability is given by:

\[
\frac{(w^{\text{LHP}})^' \sigma^L}{\| \sigma^L \|} = \frac{1}{\| \sigma^L \|} \left( \frac{\sigma^L \sigma^L - 1}{\sigma^L} \right)^' \times \left( \frac{1}{\sigma^L} \right)^' \| \sigma^L \| \\
= \frac{\| \sigma^L \|}{\| \sigma^L \|} \times \text{sign} \left( \frac{\sigma^L \sigma^L}{\sigma^L} \right),
\]

the absolute value of which is the right side of (4.1).
A similar computation shows that the correlation between strategy (4.2) and liability is equal to (4.1). If \( \omega \) denotes any other portfolio strategy, we have, by definition of the LHP, that the squared correlation between the wealth generated by \( \omega \) and liability, is lower than the squared correlation between LHP and liability:

\[
\left( \frac{w'_i \sigma' \sigma^L}{\| \sigma^L \| \sqrt{w'_i \sigma' \sigma w_i}} \right)^2 \leq \left( \frac{\| \sigma (\sigma')^{-1} \sigma' \sigma^L \|}{\| \sigma^L \|} \right)^2.
\]

Taking square roots of both sides, we obtain that the absolute correlation, hence the correlation itself, between wealth and liability, is lower than the ratio (4.1):

\[
\frac{w'_i \sigma' \sigma^L}{\| \sigma^L \| \sqrt{w'_i \sigma' \sigma w_i}} \leq \frac{w'_i \sigma' \sigma^L}{\| \sigma^L \| \sqrt{w'_i \sigma' \sigma w_i}} \leq \frac{\| \sigma (\sigma')^{-1} \sigma' \sigma^L \|}{\| \sigma^L \|}.
\]

A.5 Likelihood Maximization

In this appendix we explain how to write the joint log-likelihood of the complete observation vector (3.5). As explained in section 3, we in fact estimate the parameters by maximizing marginal likelihoods, but the technique below can be easily adapted to write these likelihoods. The continuous-time dynamics of the partially unobservable state vector \( \mathbf{V} \) can be written as:

\[
d\mathbf{V}_t = (C + A\mathbf{V}_t) \, dt + \sigma' \, dz_t,
\]

where:

\[
C = \begin{pmatrix}
ab & \pi - \frac{(\sigma^y)^2}{2} & -\frac{(\sigma^y)^2}{2} & \kappa \lambda & \sigma^y \lambda^y - \frac{(\sigma^y)^2}{2} \\
-a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & \sigma^S & 0 \\
0 & 0 & -\kappa & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
A = \begin{pmatrix}
-\sigma^r & -\sigma^F & -\sigma^S & -\sigma^\lambda & -\sigma^Y \\
-\sigma^r & -\sigma^F & -\sigma^S & -\sigma^\lambda & -\sigma^Y \\
-\sigma^r & -\sigma^F & -\sigma^S & -\sigma^\lambda & -\sigma^Y \\
-\sigma^r & -\sigma^F & -\sigma^S & -\sigma^\lambda & -\sigma^Y \\
-\sigma^r & -\sigma^F & -\sigma^S & -\sigma^\lambda & -\sigma^Y
\end{pmatrix},
\]

\[
\sigma = \begin{pmatrix}
\sigma^r & \sigma^F & \sigma^S & \sigma^\lambda & \sigma^Y
\end{pmatrix}.
\]

Denote the observation dates with \( t_0, ..., t_i = t_0 + i\Delta t, ..., t_{\nu} \). The distribution of \( \mathbf{V}_{t_{i+1}} \) conditional on \( \mathbf{V}_t \) is given by the following equality, which is obtained by integrating (A.7) from \( t_i \) to \( t_{i+1} \):

\[
\mathbf{V}_{t_{i+1}} = \Psi_0 + \Psi_1 \mathbf{V}_t + \int_{t_i}^{t_{i+1}} e^{-sA} \sigma' \, dz_t,
\]

where:

\[
\Psi_0 = \int_0^{\Delta t} e^{sA} C \, ds, \quad \Psi_1 = e^{\Delta t A}, \quad \Sigma = \mathbb{E} \left[ \mathbf{V}_{t_{i+1}} \right] = \int_0^{\Delta t} e^{sA} \sigma' \sigma e^{sA'} \, ds.
\]

We now denote with \( \tilde{\mathbf{V}} \) the modified state vector:

\[
\tilde{\mathbf{V}}_t = \begin{pmatrix}
r_t \\
\log \frac{\phi_{t_i}}{\phi_{t_{i-\Delta t}}} \\
\log \frac{S_{t_i}}{S_{t_{i-\Delta t}}} \\
\lambda_{t_i} \\
\log \frac{Y_{t_i}}{Y_{t_{i-\Delta t}}}
\end{pmatrix} = \mathbf{V}_t - MV_{t-\Delta t},
\]
with:

\[
M = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Hence we have, using \( (A.8) \):

\[
\tilde{V}_{t+1} = \Psi_0 + [\Psi_1 - M] \tilde{V}_t + [\Psi_1 - M] MV_{t+1} + \epsilon_{t+1}.
\]

In fact, the matrix \([\Phi_1 - M] M\) is zero. To see this, let us invoke the following argument. It suffices to focus on columns 2, 3 and 5, since the 1st and the 5th columns of \( M \) are zero. Let us consider the 2nd column of \([\Phi_1 - M] M\), which is equal to the 2nd column of \([\Phi_1 - M]\). Since the 2nd column of \( A \) is null, the 2nd column of \( A^k \) is also null for all \( k \geq 1 \), hence the 2nd column of \( e^{\Delta t A} \) is the vector \( (0, 1, 0, 0, 0) \). This shows that the 2nd column of \([\Phi_1 - M]\) is zero, hence that \([\Phi_1 - M]\) \( M = 0 \). We thus conclude that the modified state vector evolves as:

\[
\tilde{V}_{t+1} = \Psi_0 + \tilde{\Psi}_1 V_t + \epsilon_{t+1},
\]

with \( \tilde{\Psi}_1 = \Psi_1 - M \). Equation \( (A.9) \) is the transition equation: it shows that conditional on \( \tilde{V}_t \), the state vector \( V_{t+1} \) is multivariate normally distributed.

We denote with \( Z_t \) the vector that contains the perfectly observed yield and the logarithmic returns on the price index, the stock index and the real asset value, and the dividend yield:

\[
Z_t = \begin{pmatrix}
y(t, 3M) & \tilde{y}(t, 1Y) & \tilde{y}(t, 3Y) & \tilde{y}(t, 5Y) & \tilde{y}(t, 10Y) \\
\log \frac{S_t}{S_{t-\Delta t}} & \log \frac{S_t}{S_{t-\Delta t}} & \text{DY}_t & \log \frac{V_t}{V_{t-\Delta t}}
\end{pmatrix}^	op.
\]

Computing explicitly the expected return on the stock in our model, we obtain:

\[
\mathbb{E}_t \left[ \log \frac{S_{t+\Delta t}}{S_t} \right] = \left[ b + \sigma^2 \bar{\lambda} - \frac{(\sigma^2)^2}{2} \right] \Delta t + D(\Delta t)(r_t - b) + \sigma^2 \frac{1 - e^{-\kappa \Delta t}}{\kappa} (\lambda^S - \bar{\lambda}),
\]

while the return on the T-bill is:

\[
(D(\Delta t) y(t, \Delta t) = D(\Delta t)(r_t - C(\Delta t)).
\]

Assumption (3.2) therefore implies that \( \lambda^S_t \) itself is an affine function of the dividend yield:

\[
\tilde{m} = \frac{m}{\sigma^S \frac{1 - e^{-\kappa \Delta t}}{\kappa}}, \quad \tilde{p} = \bar{\lambda} + \frac{p - C(\Delta t) + D(\Delta t)b - \left[ b + \sigma^S \bar{\lambda} - \frac{(\sigma^2)^2}{2} \right] \Delta t}{\sigma^S \frac{1 - e^{-\kappa \Delta t}}{\kappa}}.
\]

We thus have that:

\[
Z_{t_1} = F_1 \tilde{V}_{t_1} + F_0 + \begin{pmatrix}
0 \\
\eta_t \\
0 \\
0 \\
0
\end{pmatrix},
\]

\[
(A.11)
\]
where:

$$F_1 = \begin{pmatrix}
\frac{D(\Delta t)}{\Delta t} & 0 & 0 & 0 \\
\frac{D(1)}{\Delta t} & 0 & 0 & 0 \\
\frac{D(3)}{3} & 0 & 0 & 0 \\
\frac{D(5)}{5} & 0 & 0 & 0 \\
\frac{D(10)}{10} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{1}{m} \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad F_0 = \begin{pmatrix}
-\frac{C(\Delta t)}{\Delta t} & 0 & 0 & 0 \\
-\frac{C(1)}{1} & 0 & 0 & 0 \\
-\frac{C(3)}{3} & 0 & 0 & 0 \\
-\frac{C(5)}{5} & 0 & 0 & 0 \\
-\frac{C(10)}{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{\bar{p}}{m} & 0 & 0 & 0
\end{pmatrix}, \quad \eta_t = \begin{pmatrix}
\eta_{t1}^S \\
\eta_{t2}^S \\
\eta_{t3}^S \\
\eta_{t4}^S \\
\eta_{t5}^S
\end{pmatrix}.$$
Hence the discrete-time dynamics of $W$:

$$U_{t+1} = \Psi_0^U + \Psi_1^U U_t + \int_{t}^{t+1} e^{-sA^U} (\sigma^U)' d\mathbf{z}_t.$$ 

We also define the modified state vector $\tilde{U}$ as:

$$\tilde{U}_{t_i} = \begin{pmatrix} r_{t_i} & \log \frac{S_{t_i}}{S_{t_{i-1}}} & \chi_{t_i}^u \end{pmatrix}',
= U_{t_i} - M^U U_{t_{i-1}},$$

where:

$$M^U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The discrete-time dynamics of $\tilde{U}$ is therefore given by:

$$\tilde{U}_{t+1} = \tilde{\Psi}_0^U + \tilde{\Psi}_1^U \tilde{U}_t + \varepsilon_{t+1}^U,$$  \hspace{1cm} \text{(A.13)}

where:

$$\tilde{\Psi}_1^U = \Psi_1^U - M^U, \quad \forall \varepsilon_{t+1}^U = \int_0^{\Delta t} e^{sA^U} (\sigma^U)' \sigma^U e^{s(A^U)'} ds \equiv \Sigma^U.$$

The excess return on the stock index over the T-bill over the period $[t_{i-1}, t_i]$ can then be written as:

$$Z_{t_i} \overset{\text{def}}{=} \log \frac{S_{t_i}}{S_{t_{i-1}}} - (\Delta t) y(t_{i-1}, \Delta t) = M_1 \tilde{U}_{t_i} + M_2 \tilde{U}_{t_{i-1}} + C(\Delta t),$$

with:

$$M_1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad M_2 = -D(\Delta t) \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} \text{(A.14)}

Hence:

$$\mathcal{V}_{t_0} \left[ \sum_{i=1}^{n} Z_{t_{i-1}} \right] = M_1 \mathcal{V}_{t_0} \left[ \sum_{i=1}^{n} \tilde{U}_{t_{i-1}} \right] M_1' + M_2 \mathcal{V}_{t_0} \left[ \sum_{i=1}^{n-1} \tilde{U}_{t_i} \right] M_2'
+ M_1 \text{Cov}_{t_0} \left[ \sum_{i=1}^{n} \tilde{U}_{t_{i-1}}, \sum_{i=1}^{n} \tilde{U}_{t_i} \right] M_1.'$$

We thus have to compute the covariance matrix of $\sum_{i=1}^{n} \tilde{U}_{t_i}$, and the covariance matrix of the vectors $\sum_{i=1}^{n} \tilde{U}_{t_i}$ and $\sum_{i=1}^{n-1} \tilde{U}_{t_i}$. These covariances can be obtained from the transition equation \text{(A.13)}. It can be shown that:

$$\mathcal{V}_{t_0} \left[ \sum_{i=1}^{n} \tilde{U}_{t_i} \right] = \sum_{i=0}^{k} \sum_{i=0}^{k} \left( \tilde{\Psi}_1^U \right)^i \Sigma^U \left[ \sum_{i=0}^{k} \left( \tilde{\Psi}_1^U \right)^i \right]'$$

and that:

$$\text{Cov}_{t_0} \left[ \sum_{i=1}^{n} \tilde{U}_{t_{i-1}}, \sum_{i=1}^{n} \tilde{U}_{t_i} \right] = \sum_{k=1}^{n-1} \sum_{i=0}^{k-1} \left( \tilde{\Psi}_1^U \right)^i \Sigma^U \left[ \sum_{i=0}^{k} \left( \tilde{\Psi}_1^U \right)^i \right]' \text{'}. $$
A.7 Model-Implied Correlations with Liability

We have to justify equations (4.3) to (4.7). Let us start with the stock index, whose logarithm evolves as:

\[ \text{d} \log S_t = \left[ r_t + \sigma^S \lambda_t^S - \frac{(\sigma^S)^2}{2} \right] \text{d}t + (\sigma^S)' \text{d}z_t. \]

Moreover, the dynamics of the log price of the zero-coupon bond of maturity \( T \) is:

\[ \text{d} \log B(t, T) = \left[ r_t - D(T - t)\sigma^r \lambda^r - \frac{||\sigma^B(T - t)||^2}{2} \right] \text{d}t + \sigma^B(T - t)' \text{d}z_t. \]  

Hence:

\[ \text{d} \log \frac{S_t}{B(t, T)} = \left[ \sigma^S \lambda_t^S - \frac{(\sigma^S)^2}{2} + D(T - t)\sigma^r \lambda^r + \frac{||\sigma^B(T - t)||^2}{2} \right] \text{d}t \]

\[ + \left[ \sigma^S - \sigma^B(T - t) \right]' \text{d}z_t. \]

Integrating from 0 to \( T \), we obtain:

\[ \log \frac{S_T}{S_0} = \log \frac{S_0}{B(0, T)} + \int_0^T \left[ -\frac{(\sigma^S)^2}{2} + D(T - t)\sigma^r \lambda^r + \frac{||\sigma^B(T - t)||^2}{2} \right] \text{d}t \]

\[ + \sigma^S \int_0^T \lambda_t^S \text{d}t + \int_0^T \left[ \sigma^S - \sigma^B(T - t) \right]' \text{d}z_t. \]  

Integrating the \( \lambda_t^S \) in equation (2.6), we also obtain:

\[ \int_0^T \lambda_t^S \text{d}t = \lambda T - \frac{1 - e^{-\lambda T}}{\kappa} (\lambda\lambda_0) + \frac{1}{\kappa} \int_0^T \left( 1 - e^{-\kappa(T - t)} \right) \left( \lambda \lambda_0 \right) \text{d}t. \]

Plugging this expression back into (A.16), we obtain:

\[ \log \frac{S_T}{S_0} = \log \frac{S_0}{B(0, T)} + \int_0^T \left[ -\frac{(\sigma^S)^2}{2} + D(T - t)\sigma^r \lambda^r + \frac{||\sigma^B(T - t)||^2}{2} \right] \text{d}t \]

\[ + \sigma^S \left[ \lambda T - \frac{1 - e^{-\lambda T}}{\kappa} (\lambda\lambda_0) \right] + \int_0^T \left[ \sigma^S - \sigma^B(T - t) + \sigma S_s - s^B(T - t) \frac{1 - e^{-\kappa(T - t)}}{\kappa} \lambda \lambda_0 \right]' \text{d}z_t. \]

Letting \( F(T) \) denote the sum of the first three terms in the right side, we obtain (4.4).

For other asset classes, we can proceed in a similar way, but the equations will become slightly simpler because the Sharpe ratios of these assets are constant. For instance, for the constant-maturity nominal bond, we have:

\[ \log \frac{B_T}{B_0} = \log \frac{B_0}{B(0, T)} + \int_0^T \left[ [D(T - t) - D(\tau_B)] \sigma^r \lambda^r - \frac{D(\tau_B)^2}{2} (\sigma^r)^2 + \frac{||\sigma^B(T - t)||^2}{2} \right] \text{d}t \]

\[ + \int_0^T \left[ \sigma^B(T - t) - \sigma^B(T - t) \right]' \text{d}z_t, \]

which is of the form (4.6).

We now derive correlations at horizon \( T \) between LHPs and liabilities. We consider for example a LHP invested only in nominal bonds and one real asset class. Then the volatility matrix and the market price of risk vector are constant and given by:

\[ \sigma = \left( \sigma^B(\tau_B) \quad \sigma^r \right), \quad \lambda_t = \sigma(\sigma')^{-1} \left( \frac{-D(\tau_B)\sigma^r \lambda^r}{\sigma^r \lambda^r} \right). \]
Hence the dynamics of the log wealth when strategy (4.2) is followed:

\[
\frac{\mathrm{d}\log A_t}{\mathrm{d}t} = \left[ r + w_t \mathbf{\alpha}' \mathbf{\lambda} - \frac{\|\mathbf{\sigma} w_t\|^2}{2} \right] \, \mathrm{d}t + w_t \mathbf{\alpha}' \, \mathrm{d}z_t.
\]

Subtracting (A.15) from this equation, we obtain:

\[
\frac{\mathrm{d}\log A_t}{\mathrm{d}t} = \left[ \sigma^L (\tau_L - t)' \sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\sigma}' \mathbf{\lambda} + D(T - t) \sigma' \mathbf{\lambda} \right] \, \mathrm{d}t
\]
\[
+ \frac{\|\sigma^B (T - t)\|^2 - \|\sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\alpha}' \sigma (\tau_L - t)\|^2}{2} \, \mathrm{d}t
\]
\[
+ \left[ \sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\alpha}' \sigma (\tau_L - t) - \sigma^B (T - t) \right]' \, \mathrm{d}z_t.
\]

Integrating from 0 to \( T \), we obtain the log return on wealth over \([0, T]\):

\[
\int_0^T \left[ \sigma^L (\tau_L - t)' \sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\sigma}' \mathbf{\lambda} + D(T - t) \sigma' \mathbf{\lambda} \right] \, \mathrm{d}t
\]
\[
+ \frac{\|\sigma^B (T - t)\|^2 - \|\sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\alpha}' \sigma (\tau_L - t)\|^2}{2} \, \mathrm{d}t
\]
\[
+ \int_0^T \left[ \sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\alpha}' \sigma (\tau_L - t) - \sigma^B (T - t) \right]' \, \mathrm{d}z_t.
\]

(A.17)

Hence the variance of the log return:

\[
\mathbb{V} \left[ \log \frac{A_T}{A_0} \right] = \int_0^T \left\| \sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\alpha}' \sigma (\tau_L - t) - \sigma^B (T - t) \right\|^2 \, \mathrm{d}t.
\]

(A.18)

To write the covariance between wealth and liabilities, we use equation (4.3) together with (A.17):

\[
\text{Cov} \left[ \log \frac{A_T}{A_0}, \log \frac{L_T}{L_0} \right] = \int_0^T \left[ \sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\alpha}' \sigma (\tau_L - t) - \sigma^B (T - t) \right]' \left[ \sigma^L (\tau_L - t) - \sigma^B (T - t) \right] \, \mathrm{d}t.
\]

(A.19)

In particular, if \( T = \tau_L \), equation (A.18) becomes:

\[
\mathbb{V} \left[ \log \frac{A_{\tau_L}}{A_0} \right] = \int_0^{\tau_L} \left\| \sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\alpha}' \mathbf{\phi} \right\|^2 \, \mathrm{d}t
\]
\[
\tau_L \left\| \sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\alpha}' \mathbf{\phi} \right\|^2,
\]

and (A.19) implies:

\[
\text{Cov} \left[ \log \frac{A_{\tau_L}}{A_0}, \log \frac{L_{\tau_L}}{L_0} \right] = \int_0^{\tau_L} (\mathbf{\phi}')' \sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\alpha}' \mathbf{\phi} \, \mathrm{d}t
\]
\[
\tau_L \left\| \sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\alpha}' \mathbf{\phi} \right\|^2.
\]

Moreover, (4.3) gives:

\[
\mathbb{V} \left[ \log \frac{L_{\tau_L}}{L_0} \right] = \tau_L \left\| \mathbf{\phi} \right\|^2.
\]

Hence:

\[
\text{Corr} \left[ \log \frac{A_{\tau_L}}{A_0}, \log \frac{L_{\tau_L}}{L_0} \right] = \frac{\left\| \sigma (\mathbf{\alpha}' \mathbf{\sigma})^{-1} \mathbf{\alpha}' \mathbf{\phi} \right\|}{\left\| \mathbf{\phi} \right\|}.
\]

which is equation (4.8).
B. Tables

Table 1: Maximum Likelihood estimates.

<table>
<thead>
<tr>
<th>Nominal short-term rate: $dr_t = (b - r_t) dt + \sigma^r dz^r_t$.</th>
<th>Standard deviation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>0.1428</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.0265</td>
</tr>
<tr>
<td>$\sigma^r$</td>
<td>0.0160</td>
</tr>
<tr>
<td>$\lambda^r$</td>
<td>-0.5158</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stock price process: $dS_t/S_t = \left[ r_t + \sigma^S \lambda^S \right] dt + \sigma^S dz^S_t$.</th>
<th>Standard deviation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma^S$</td>
<td>0.2125</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sharpe ratio: $d\lambda^S_t = \kappa \left( \lambda_t - \lambda^S \right) dt + \sigma^S dz^S_t$, $E_t \left[ \log \frac{S_{t+1}}{S_t} \right] - (\Delta t) y(t, \Delta t) = mDY_t + p$.</th>
<th>Standard deviation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>0.2140</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.2905</td>
</tr>
<tr>
<td>$\sigma^S$</td>
<td>0.1239</td>
</tr>
<tr>
<td>$m$</td>
<td>0.0284</td>
</tr>
<tr>
<td>$p$</td>
<td>0.1215</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Price index process: $d\Phi_t/\Phi_t = \pi dt + \sigma^\Phi dz^\Phi_t$.</th>
<th>Standard deviation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>0.0388</td>
</tr>
<tr>
<td>$\sigma^\Phi$</td>
<td>0.0188</td>
</tr>
<tr>
<td>$\lambda^\Phi$</td>
<td>0.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Real asset process: $dY_t/Y_t = \left( r_t + \sigma^Y \lambda^Y \right) dt + \sigma^Y dz^Y_t$.</th>
<th>Commodities.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma^Y$</td>
<td>0.2624</td>
</tr>
<tr>
<td>$\lambda^Y$</td>
<td>0.3462</td>
</tr>
<tr>
<td>$\sigma^Y$</td>
<td>0.2843</td>
</tr>
<tr>
<td>$\lambda^Y$</td>
<td>0.2752</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho^S\lambda$</td>
<td>0.0358</td>
<td>0.0999</td>
</tr>
<tr>
<td>$\rho^S\phi$</td>
<td>-0.0990</td>
<td>0.0100</td>
</tr>
<tr>
<td>$\rho^\lambda\phi$</td>
<td>0.1649</td>
<td>0.1128</td>
</tr>
<tr>
<td>$\rho^S\lambda$</td>
<td>-0.9293</td>
<td>0.0139</td>
</tr>
<tr>
<td>$\rho^S\phi$</td>
<td>0.1242</td>
<td>0.1130</td>
</tr>
<tr>
<td>$\rho^\lambda\phi$</td>
<td>-0.0736</td>
<td>0.1144</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho^{SY}$</td>
<td>0.5934</td>
<td>0.0552</td>
</tr>
<tr>
<td>$\rho^{GY}$</td>
<td>0.4663</td>
<td>0.1578</td>
</tr>
<tr>
<td>$\rho^{AY}$</td>
<td>-0.5473</td>
<td>0.0580</td>
</tr>
<tr>
<td>$\rho^{GY}$</td>
<td>0.2876</td>
<td>0.0962</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho^{SY}$</td>
<td>0.3803</td>
<td>0.0712</td>
</tr>
<tr>
<td>$\rho^{GY}$</td>
<td>0.1508</td>
<td>0.0971</td>
</tr>
<tr>
<td>$\rho^{AY}$</td>
<td>-0.2558</td>
<td>0.0763</td>
</tr>
<tr>
<td>$\rho^{GY}$</td>
<td>0.4184</td>
<td>0.0775</td>
</tr>
</tbody>
</table>

Volatilities of measurement errors on bond yields.

<table>
<thead>
<tr>
<th>Estimate.</th>
<th>Standard deviation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{1Y}$</td>
<td>&lt; 0.0001</td>
</tr>
<tr>
<td>$s_{2Y}$</td>
<td>&lt; 0.0001</td>
</tr>
<tr>
<td>$s_{3Y}$</td>
<td>&lt; 0.0001</td>
</tr>
<tr>
<td>$s_{10Y}$</td>
<td>&lt; 0.0001</td>
</tr>
</tbody>
</table>

Estimates for all parameter values except $\lambda^\Phi$ have been obtained by maximizing the log-likelihood of quarterly observations on Canadian and US markets. Standard deviations have been computed by taking the square roots of diagonal elements of the inverse of Fisher's information matrix.
Table 2: Decomposition of liability risk into spanned risk and unspanned risk.

<table>
<thead>
<tr>
<th>LHP</th>
<th>Time-to-maturity (years)</th>
<th>15</th>
<th>10</th>
<th>5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>96.38</td>
<td>98.17</td>
<td>95.13</td>
<td>97.54</td>
<td>89.46</td>
</tr>
<tr>
<td>S</td>
<td>&lt;0.01</td>
<td>1.23</td>
<td>&lt;0.01</td>
<td>0.01</td>
<td>&lt;0.01</td>
</tr>
<tr>
<td>Com</td>
<td>0.52</td>
<td>7.21</td>
<td>0.35</td>
<td>5.91</td>
<td>&lt;0.01</td>
</tr>
<tr>
<td>RE</td>
<td>17.40</td>
<td>41.72</td>
<td>16.62</td>
<td>40.77</td>
<td>13.82</td>
</tr>
<tr>
<td>I</td>
<td>99.90</td>
<td>99.95</td>
<td>100</td>
<td>100</td>
<td>98.83</td>
</tr>
<tr>
<td>B, S</td>
<td>96.43</td>
<td>98.2</td>
<td>95.2</td>
<td>97.57</td>
<td>89.61</td>
</tr>
<tr>
<td>B, Com</td>
<td>96.97</td>
<td>98.47</td>
<td>95.93</td>
<td>97.94</td>
<td>91.18</td>
</tr>
<tr>
<td>B, RE</td>
<td>96.59</td>
<td>98.28</td>
<td>95.42</td>
<td>97.68</td>
<td>90.08</td>
</tr>
<tr>
<td>B, S, Com</td>
<td>96.97</td>
<td>98.48</td>
<td>97.94</td>
<td>98.04</td>
<td>91.19</td>
</tr>
<tr>
<td>B, S, RE</td>
<td>96.60</td>
<td>98.29</td>
<td>95.43</td>
<td>97.69</td>
<td>90.10</td>
</tr>
<tr>
<td>B, I</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
</tr>
</tbody>
</table>

This table shows the absolute correlation between LHP and liability (column Corr.), as well as the percentage of the total variance of liability that is explained by the base assets (column % Var.). Ten versions of the LHP are considered. They are based on the following assets: nominal bonds of constant maturity 10 years (B), stocks (S), commodities (Com), real estate (RE) and indexed bonds of constant maturity 10 years (I). All parameter values are taken from table 1.

C. Figures

Zero-coupon yields are obtained from Bloomberg. The time-series for the short-term rate has been filtered out from zero-coupon yields by Likelihood Maximization, assuming that the 3-month yield is observed without error.
Figure 2: Stochastic Sharpe ratio of stock index.

On subfigure (a), the instantaneous Sharpe ratio of the S&P500 index (expressed in Canadian $) is computed from (3.3). The dividend yield is computed from CRSP data. The value of S&P500 with dividends reinvested is also obtained from CRSP and is converted to C$. Its value is taken equal to C$ 1 on January 1st, 1966. Subfigure (b) shows the annualized volatility of stock returns at different investment horizons, as implied by the model of section 2.
Figure 3: Term structures of correlation of asset classes with liabilities.

(a) Stocks.

(b) 10-year nominal bonds.

(c) Real estate.

(d) Commodities.

(e) 10-year inflation-indexed bonds.

The figure displays the correlations of nominal returns on five asset classes with liabilities at different horizons. These correlations are computed from the parameters of table 1. The liability portfolio is a fixed-maturity inflation-indexed bond with initial maturity \( L \) equal to 1, 5, 10 or 15 years.
The figure displays the correlations of nominal returns on five asset classes with realized inflation at different horizons. The asset classes considered are 10-year constant-maturity nominal bonds (B), stocks (S), 10-year constant-maturity indexed bonds (I), real estate (RE) and commodities (Com). These correlations are computed from the parameters of table 1.

This figure shows the model-implied correlations between the wealth generated by a strategy that fully invests in the LHP and the cash (see equation (4.2)), and the liabilities. The liability portfolio is a fixed-maturity inflation-indexed bond, with initial maturity $\tau_L$ equal to 1, 5, 10 or 15 years. The LHP is computed over one of the following four investment universes: nominal bonds and stocks (B, S); nominal bonds and real estate (B, RE); nominal bonds and commodities (B, Com); nominal bonds and inflation-indexed bonds (B, I). All parameters are set to the calibrated values [see table 1]
Utility cost is the fraction by which initial wealth must be increased for the investor to be willing to replace a perfect LHP by an imperfect LHP made of stocks and nominal bonds (solid lines) or by an imperfect LHP made of nominal bonds only (dotted lines). All these utility costs are expressed as fractions of the initial wealth $A_0$, and they are independent of $A_0$. We consider three levels of risk aversion $\gamma$. The liability portfolio is a fixed-maturity indexed bond of initial maturity $\tau_L$ equal to 1, 5, 10 or 15 years, and we let the investment horizon vary from one quarter to $\tau_L$. Other parameters are set to the calibrated values (see table 1), and the initial values $r_0$ and $\lambda_0$ are taken equal to the long-term means $b$ and $\bar{\lambda}$. 

Figure 6: Utility cost of giving up indexed bonds in the liability-hedging portfolio.
Utility cost is the fraction by which initial wealth must be increased for the investor to be willing to ignore completely predictability in stock returns when investing in stocks, nominal bonds and cash. The suboptimal strategy consists therefore of assuming a constant Sharpe ratio for stocks and of ignoring the hedging demand against time variation in this Sharpe ratio. All utility costs are expressed as fractions of the initial wealth $A_0$, and they are independent of $A_0$. We consider three levels of risk aversion $\gamma$. The liability portfolio is a fixed-maturity indexed bond of initial maturity $\tau_L$ equal to 1, 5, 10 or 15 years, and we let the investment horizon vary from one quarter to $\tau_L$. Other parameters are set to the calibrated values (see table 1), and the initial values $r_0$ and $\lambda_0$ are taken equal to the long-term means $b$ and $\lambda$. 
Utility cost is the fraction by which initial wealth must be increased for the investor to be willing to replace an imperfect LHP of nominal bonds, stocks and commodities by an imperfect LHP of nominal bonds and stocks only. All these utility costs are expressed as fractions of the initial wealth $A_0$, and they are independent of $A_0$. We consider three levels of risk aversion $\gamma$. The liability portfolio is a fixed-maturity indexed bond of initial maturity $\tau_L$ equal to 1, 5, 10 or 15 years, and we let the investment horizon vary from one quarter to $\tau_L$. Other parameters are set to the calibrated values (see table 1), and the initial values $r_0$ and $\lambda_0$ are taken equal to the long-term means $b$ and $\bar{\lambda}$. 

Figure 8: Utility cost of giving up commodities in the liability-hedging portfolio.
Utility cost is the fraction by which initial wealth must be increased for the investor to be willing to replace an imperfect LHP of nominal bonds, stocks and real estate by an imperfect LHP of nominal bonds and stocks only. All these utility costs are expressed as fractions of the initial wealth $A_0$, and they are independent of $A_0$. We consider three levels of risk aversion $\gamma$. The liability portfolio is a fixed-maturity indexed bond of initial maturity $\tau_L$ equal to 1, 5, 10 or 15 years, and we let the investment horizon vary from one quarter to $\tau_L$. Other parameters are set to the calibrated values (see table 1), and the initial values $r_0$ and $\lambda_0$ are taken equal to the long-term means $\bar{b}$ and $\bar{\lambda}$. 

Figure 9: Utility cost of giving up real estate in the liability-hedging portfolio.
Utility cost is the fraction by which initial wealth must be increased for the investor to be willing to give up commodities if he has the opportunity to invest in nominal bonds, stocks, commodities and cash. All these utility costs are expressed as fractions of the initial wealth $A_0$ and they are independent of $A_0$. We consider three levels of risk aversion $\gamma$. The liability portfolio is a fixed-maturity indexed bond of initial maturity $\tau_L$ equal to 1, 5, 10 or 15 years, and we let the investment horizon vary from one quarter to $\tau_L$. Other parameters are set to the calibrated values (see table 1), and the initial values $r_0$ and $\bar{\sigma}$ are taken equal to the long-term means $\bar{b}$ and $\bar{\lambda}$. 

![Figure 10: Utility cost of giving up commodities in the optimal allocation.](image)

(a) Liability maturity 1 year.

(b) Liability maturity 5 years.

(c) Liability maturity 10 years.

(d) Liability maturity 15 years.
Utility cost is the fraction by which initial wealth must be increased for the investor to be willing to give up real estate if he has the opportunity to invest in nominal bonds, stocks, real estate and cash. All these utility costs are expressed as fractions of the initial wealth $A_0$, and they are independent of $A_0$. We consider three levels of risk aversion $\gamma$. The liability portfolio is a fixed-maturity indexed bond of initial maturity $\tau_L$ equal to 1, 5, 10 or 15 years, and we let the investment horizon vary from one quarter to $\tau_L$. Other parameters are set to the calibrated values (see table 1), and the initial values $r_0$ and $\lambda_0$ are taken equal to the long-term means $b$ and $\bar{\lambda}$. 

Utility cost of giving up real estate in the optimal allocation.
References


Founded in 1906, EDHEC Business School offers management education at undergraduate, graduate, post-graduate and executive levels. Holding the AACSB, AMBA and EQUIS accreditations and regularly ranked among Europe’s leading institutions, EDHEC Business School delivers degree courses to over 6,000 students from the world over and trains 5,500 professionals yearly through executive courses and research events. The School’s ‘Research for Business’ policy focuses on issues that correspond to genuine industry and community expectations.

Established in 2001, EDHEC-Risk Institute has become the premier academic centre for industry-relevant financial research. In partnership with large financial institutions, its team of ninety permanent professors, engineers, and support staff, and forty-eight research associates and affiliate professors, implements six research programmes and sixteen research chairs and strategic research projects focusing on asset allocation and risk management. EDHEC-Risk Institute also has highly significant executive education activities for professionals. It has an original PhD in Finance programme which has an executive track for high level professionals. Complementing the core faculty, this unique PhD in Finance programme has highly prestigious affiliate faculty from universities such as Princeton, Wharton, Oxford, Chicago and CalTech.

In 2012, EDHEC-Risk Institute signed two strategic partnership agreements with the Operations Research and Financial Engineering department of Princeton University to set up a joint research programme in the area of risk and investment management, and with Yale School of Management to set up joint certified executive training courses in North America and Europe in the area of investment management.

For more information, please contact:
Carolyn Essid on +33 493 187 824
or by e-mail to: carolyn.essid@edhec-risk.com