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Abstract
This paper introduces a novel method for the construction of equity indices that, unlike their cap-weighted counterparts, offer an efficient risk/return tradeoff. The index construction method goes back to the roots of modern portfolio theory and focuses on the tangency portfolio, the portfolio that weights index constituents so as to obtain the highest possible Sharpe ratio. The major challenge is to generate the required input parameters in a robust manner. The expected excess return of each stock is estimated from portfolio sorts according to the stock’s total downside risk. This estimate uses the economic insight that stocks with higher risk should compensate their holders with higher expected returns. To estimate the covariance matrix, we use principal component analysis to extract the common factors driving stock returns. Moreover, we introduce a procedure to control turnover in order to implement the method with low transaction costs. Our empirical results show that portfolio optimisation with our robust parameter estimates generates out-of-sample Sharpe ratios significantly higher than those of the corresponding cap-weighted indices. In addition, the higher risk/return efficiency is achieved consistently and across varying economic and market conditions.
Introduction
Introduction

The capital asset pricing model (CAPM), introduced by Sharpe (1964), has had a profound influence on the management of institutional portfolios. The CAPM starts with a series of assumptions and theorises that the market portfolio of all assets is risk/return efficient in the sense that it provides the highest possible expected return above the risk-free rate per unit of volatility, i.e., the highest Sharpe ratio. Since the CAPM is taught in business schools around the world, there is a widespread belief that all investors should hold the market portfolio, leveraged or de-leveraged with positions in the risk-free asset depending on their risk aversion. In practice, "holding the market" becomes virtually impossible, but its approximate implementation in terms of some market-capitalisation-weighted equity indices has become the standard practice for most investors and asset managers.

Capitalisation weighting in equity index construction has, however, come in for harsh criticism. Early papers by Haugen and Baker (1991) or Grinold (1992) provide empirical evidence that market-cap-weighted indices provide an inefficient risk/return tradeoff. From the theoretical standpoint, the poor risk-adjusted performance of such indices should come as no surprise, as market-cap-weighting schemes are risk/return efficient only at the cost of heroic assumptions.

• The theoretical basis for holding the market portfolio is the CAPM. An extensive body of literature has shown that the theoretical prediction of an efficient market portfolio breaks down when some of the highly unrealistic assumptions of the CAPM do not bear out. In particular, financial theory does not predict that the market portfolio is efficient if investors have different time horizons, if they derive wealth from non-traded assets such as housing, social security benefits, or human capital, if short sales are constrained or if frictions in the form of taxes exist. Unsurprisingly, when testing the CAPM on securities data, the model is commonly rejected.1

• In addition, even if the CAPM were the true asset pricing model, holding a market-cap-weighted equity index would be a rather poor proxy for holding the market portfolio, which in principle is a combination of all assets, traded or non-traded, financial or non-financial, including human capital.

In the wake of criticism of market-cap-weighted indices, alternative weighting schemes have been introduced. In pursuit of a more representative weighting scheme, researchers have proposed to weight stocks by firm characteristics such as earnings, dividends, or book value (Arnott, Hsu, and Moore 2005; Siegel, Schwartz, and Siracusano 2007). Other research has focused on constructing minimum variance benchmarks (Chan, Karceski, and Lakonishok 1999; Clarke, De Silva, and Thorley 2006), maximum diversification benchmarks (Choueifaty and Coignard 2008), equal-risk contribution benchmarks (Maillard, Roncalli, and Teiletche 2008) and risk factor benchmarks (Lee 2003; Eggins and Hill 2008; Wagner and Stocker 2009).

For a rational investor, the goal is not to have the most representative portfolio or the portfolio with the lowest risk; it is instead to hold a portfolio that achieves the highest risk-adjusted performance. In the end, if investors care about a portfolio’s risk-adjusted performance, one should focus

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1 - See Goltz and Le Sourd (2010) for a literature review.
on designing a portfolio with the highest reward-to-risk ratio, i.e., with the highest Sharpe ratio. This portfolio is known as the tangency portfolio. Following Markowitz (1952), Tobin (1958) notes that any investor can separate his investment decisions into two steps. First, find the tangency portfolio and then use an investment in the risk-free asset to obtain an overall portfolio that corresponds to the investor's risk aversion. Our approach is to focus on the design of this tangency portfolio. We thus return to the roots of modern portfolio theory to provide an alternative to the current methods of constructing equity indices. The aim of this efficient indexation approach is to provide investors with benchmarks that reflect the possible risk/reward ratio from a broadly diversified stock market portfolio, and that are thus a proxy for the normal returns of an exposure to equity risk.

To generate the tangency portfolio, we resort to standard mean-variance optimisation. Although our aim to maximise risk/return efficiency is fully consistent with financial theory, successful implementation of the theory depends not only on its conceptual grounds but also on the reliability of the input to the model. In our case, the results depend greatly on the quality of the parameter estimate (the covariance matrix and the expected returns of all stocks in the index).

The CAPM, as it happens, is a poor guide to the input parameters. For the CAPM, expected returns should be proportional to the stock's beta, though it has in fact been shown that such a relationship does not hold (Fama and French 1992). Likewise, the single-factor nature of the CAPM would mean that there is a single (market) factor driving the correlation of stocks, whereas the consensus in both academe and business is that multifactor models do a better job capturing the common drivers behind stock comovements.

Extending the preliminary tests reported in Martellini (2008), we generate proxies for tangency portfolios that rely on robust input parameters for both the covariance matrix and expected returns. One challenge is the estimation of expected return parameters. Instead of relying purely on statistics, which is known to generate poor expected return estimates, we use a common sense estimate of expected returns that relies on a risk/reward tradeoff. We use the insight that the return on a given stock in excess of the risk-free rate is proportional to the riskiness of the stock. Investors are often underdiversified and averse not only to systematic risk but also to the specific risk of a stock. Investors shun the volatility, negative skewness, and kurtosis of a stock’s returns. We use a suitably designed risk measure that integrates these aspects and estimate expected returns by sorting stocks into high risk and low risk categories. The second central ingredient in the tangency portfolio is an estimate of the covariance of stock returns. We use a robust estimation procedure that first extracts the common factors of stock returns and then uses these factors to model the comovement of individual stocks. This efficient indexation procedure allows us to construct proxies for the tangency portfolios whose risk/reward ratio is significantly better than those of cap-weighted indices.

We use constituent data for the S&P 500 index to construct tangency portfolio...
proxies based on the same set of stocks as this cap-weighted index. Overall, our efficient indices obtain both higher average returns and lower volatility than do cap-weighted indices. However, portfolios rebalanced every quarter are subject to high turnover. We reduce turnover by limiting rebalancing; only when significant new information arrives do we rebalance our optimal weights.

This approach leads to significantly less turnover yet maintains high Sharpe ratios. Annual turnover in excess of the cap-weighted index is less than 20%. Over the long term, our indices increase the Sharpe ratio of the S&P 500 cap-weighted index by more than 70%. Interestingly, this improved risk/reward tradeoff does not come at the cost of an increase in extreme risks, and it holds when conditioning on business cycles or implied volatility. When performance over several ten-year periods is analysed, the efficient indexation strategy had lower Sharpe ratios only during the bull markets of the 1990s, although volatility was still lower than that of the cap-weighted indices.

Cap-weighted indices weight stocks by the footprint they leave on the stock market. Characteristics-based indices weight stocks by their footprint in the economy. Investors probably care little about these aspects, unless they want portfolios representative of the stock market or the economy. Our approach weights stocks by their “risk/return footprint” on the investor’s portfolio. Investors, of course, would prefer high weights in stocks that contribute positively to the portfolio’s Sharpe ratio and low weights in stocks that contribute less to increasing the Sharpe ratio. The contribution of this paper is to provide an index construction method that explicitly takes into account this investor objective.

The remainder of this paper is organised as follows. In section 1, we describe the parameter estimates used in the portfolio optimisation, namely, the covariance matrix and the expected returns. Section 2 is an overview of the implementation of the method, addressing issues such as data, timing, weight constraints, and turnover control. Section 3 analyses the performance of the resulting portfolios both over the long run and in terms of consistency over time and across different market conditions. A final section concludes.
1. Robust Estimation of Return Comovements and Expected Returns
1. Robust Estimation of Return Comovements and Expected Returns

A key to providing truly efficient equity indices is, first, to recognise this objective explicitly in the index construction process. However, improvement of the objective function is possible only if input parameters are reliable. We now turn to describing the derivation of input parameters, first for the covariance matrix and then for the expected returns.

1.1. Improved Estimation of the Comovements of Stock Returns

Several improved estimates for the covariance matrix have been proposed, including the factor-based approach (Sharpe 1964), the constant correlation approach (Elton and Gruber 1973), and the statistical shrinkage approach (Ledoit and Wolf 2004). In addition, Jagannathan and Ma (2003) find that imposing (short selling) constraints on the weights in the optimisation program improves the risk-adjusted out-of-sample performance in a manner similar to some of the improved covariance matrix estimators.

In these papers, the authors focus on testing the out-of-sample performance of global minimum variance (GMV) portfolios, as opposed to the tangency portfolios, as there is a consensus that available estimates of expected returns are not robust enough to be used.

The key problem in covariance matrix estimation is the curse of dimensionality; when a large number of stocks is considered, the number of parameters to estimate grows exponentially. Furthermore, the sample covariance matrix will be non-invertible if the number of assets \( N \) exceeds the number of available observations \( T \); this is particularly disturbing since the minimum variance (MV) investor’s optimal portfolio depends on the inverse of the covariance matrix.

Therefore, at the estimation stage, the challenge is to reduce the number of factors. In general, a multifactor model decomposes the (excess) return (in excess of the risk-free asset) on an asset into its expected rewards for exposure to the “true” risk factors. The use of a multifactor model originates in Ross’s (1976) arbitrage pricing theory (APT). Formally, the returns on an asset are governed by the following linear factor model:

\[
R_i = \alpha_i + \beta_i F + \varepsilon_i
\]

or in matrix form for all \( N \) assets

\[
R = \alpha + \beta F + \varepsilon
\]  \hspace{1cm} (1)

where \( \beta_i \) is an \( N \times K \) matrix containing the sensitivities of each asset \( i \) with respect to the corresponding \( j \)-th factor movements; \( \varepsilon_i \) is the vector of the \( N \) assets’ (excess) returns, \( F \) a vector containing the \( K \) risk factors’ (excess) returns, and \( \varepsilon \) the \( N \times 1 \) vector containing the zero mean uncorrelated residuals \( \varepsilon_i \). The covariance matrix for the asset returns, implied by a factor model, is given by

\[
\Omega = \beta \cdot \Sigma_F \cdot \beta^T + \Sigma_r
\]  \hspace{1cm} (2)

where \( \Sigma_F \) is the \( K \times K \) covariance matrix of the risk factors and \( \Sigma_r \) an \( N \times N \) covariance matrix of the residuals corresponding to each asset.

1.1.1. Choosing the appropriate factors

Although the factor-based estimator is expected to allow a reasonable tradeoff between sample risk and model risk, the problem of choosing the “right” factor model...
remains. We take a somewhat agnostic approach to this question, and aim to rely as little as possible on strong theoretical assumptions by using principal component analysis (PCA) to determine the underlying risk factors from the data. The PCA method is based on spectral decomposition of the sample covariance matrix and its goal is to explain covariance structures using only a few linear combinations of the original stochastic variables that will constitute the set of (unobservable) factors.

We can use PCA in the context of a factor model, making the assumption that all stock returns depend on a number of underlying and unobservable stochastic factors $F_1, F_2, \ldots F_K$, as well as on the variable specific errors/variations $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_N$. Consider the $N$-dimensional stochastic (demeaned) vector to be any of the stochastic variables $r_t$ for $t = 1, 2, \ldots, T$ with sample covariance matrix $S$.

The factor model in matrix form would be:

$$ r = LF + \varepsilon \quad (3) $$

where the coefficients $l_{ij}$ of $L$ correspond to the loading on variable $i$ by factor $j$ and $F$ is a vector with the unobservable underlying factors. Equation (3) corresponds to equation (1) assuming zero intercept (from a pricing theory standpoint, this should be valid if we have a correct factor model). We also assume that:

$$ \mathbb{E}[\varepsilon] = 0 \quad \text{Var}[\varepsilon] = \mathbb{E}[\varepsilon \varepsilon^T] = \Psi \quad (4) $$

where $\Psi$ is a diagonal matrix of specific variances in which the factors and the specific variances are meant to be uncorrelated. Letting the covariance matrix of the factors be $\Lambda$, and taking the variance of (3) gives:

$$ \text{Var}[r] = \Gamma \Lambda \Gamma^T + \Psi \quad (5) $$

The principal components are those linear combinations that give the direction of maximum variance in the sample such that they are uncorrelated with each other (orthogonal). The $i$th principal component is given by:

$$ f_i = \Gamma^T r = \sum_{n} l_{in} r_n \quad i = 1, 2, \ldots N \quad (5) $$

for which variances and covariances are:

$$ \text{Var}[f_i] = l_i^T S l_i = \lambda_i \quad i = 1, 2, \ldots N \quad (6) $$

$$ \text{Cov}[f_i, f_j] = l_i^T S l_j \quad i \neq j \quad (7) $$

and $l_i^T = 1 i = 1, 2, \ldots N$. The loadings are determined by the eigenvectors of $S$ in equation (5) and their variances in (6) equal the corresponding eigenvalues $\lambda_i$. The application of this procedure using standardised returns in $r$ make $S$ (in this notation) the correlation matrix. For clarity we use $P$ to denote the correlation matrix.

Taking the eigenvalues-eigenvector pairs $(e_1, \lambda_1), (e_2, \lambda_2), \ldots (e_N, \lambda_N)$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$ and $e_j = [e_{j1}, e_{j2}, \ldots, e_{jN}]^T$ of the matrix $P$, we can rewrite it as:

$$ P = \lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T + \ldots + \lambda_N e_N e_N^T = \Gamma \Lambda \Gamma^T $$

The decomposition in (6) fits exactly into equation (4), taking $\Psi = 0$, and noting that $\text{Cov}[f_i, f_j] = 0$. This form yields an exact representation of the covariance structure; however, a great deal of the variability can often be explained by only a few of the principal components without losing much information.

The advantage of this procedure is that it can lead to a very significant reduction of the number of parameters to estimate. This can be implemented by neglecting the effect of the smallest eigenvalues;
1. Robust Estimation of Return Comovements and Expected Returns

hence, we can write equation (6) as:

\[ P = \lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T + \ldots + \lambda_k e_k e_k^T \]

where \( L \) is an \( N \times K \) matrix with \( K < N \). If we now take into account the effect of the error embedded in the approximation we get:

\[ P = LA\Lambda^T + \Psi \]  

This is equivalent to obliging the diagonal elements in the correlation matrix to be equal to one. Note the correspondence of equations (8) and (2); both use a factor model to decompose the matrix, but equation (8) corresponds to a correlation matrix given that we take \( r \) as standardised returns (zero mean and unit variance).

Bengtsson and Holst (2002) and Fujiwara et al. (2006) also provide justification for the use of PCA in a similar way, extracting principal components to estimate expected correlation within MV portfolio optimisation. Fujiwara et al. (2006) find that the realised reward-to-risk ratio of portfolios based on the PCA method is higher than that of the single-index and that the optimisation gives a practically reasonable asset allocation. Overall, the main strength of the PCA approach at this stage is that it enables the data to talk and to show the underlying risk factors that govern most of the variability of the assets at each point in time. The PCA approach stands in great contrast to forced reliance on the assumption that a particular factor model is the true pricing model and reduces the specification risk embedded in the factor-based approach, while keeping the sample risk reduction.

Furthermore, to reduce the specification risk to the minimum, we use an objective criterion to determine the number of factors in our estimation.

1.1.2. Choosing the appropriate number of factors

Determining the appropriate number of factors to structure the correlation matrix is critical to the risk estimation when using PCA as a factor model. Several options, some with more theoretical justification than others, have been proposed to make this determination.

Financial applications such as those of Laloux et al. (1999), Bengtsson and Holst (2002), Amenc and Martellini (2002), and Fujiwara et al. (2006) provide justification for the use of a rule derived from some explicit results from random matrix theory (RMT) (Plerou et al. 2001; Plerou et al. 1999; Laloux et al.; 1998; Guhr 2001; Marchenko and Pastur 1967). Within the correlation matrix structure context, Fujiwara et al. (2006) find that the error in the estimation of the correlation matrix via RMT is more stable and smaller than that of the sample, single-index, or constant-correlation model.

The idea is to try to separate the real correlation from the estimation error by comparing the properties of the empirical correlation matrix with known results for a completely random correlation matrix. It has been shown that the asymptotic density of eigenvalues of the correlation matrix of strictly independent asset reads:

\[ f(\lambda) = \frac{T}{2N\pi} \sqrt{\lambda - \lambda_{\text{max}}}(\lambda - \lambda_{\text{min}}) \]

3 See Plerou et al. (2001).
1. Robust Estimation of Return Comovements and Expected Returns

where $\lambda_{min} \leq \lambda \leq \lambda_{max}$

and the minimum and the maximum eigenvalue bounds are given by:

$$\lambda_{max} = 1 + \frac{N}{T} \pm 2\sqrt{\frac{N}{T}}$$

(10)

A conservative interpretation of this result to design a systematic decision rule is to regard as statistical noise all factors associated with an eigenvalue lower than $\lambda_{min}$.

1.2. Improved Estimators of Expected Returns

Although we rely on statistics to extract meaningful factor models for covariance estimation, they are nearly useless in estimating expected returns, since the data are extremely noisy (Britten-Jones 1999). Recognising the difficulty of using sample-based expected return estimates in portfolio optimisation, we follow Martellini (2008) in using an economic relation to estimate expected returns. In particular, we use an estimate of the stock’s risk to proxy for a stock’s expected returns. This approach is based on the principle that investors expect an additional return for taking on more risk.

Although it seems reasonable to assume that there is a risk/return tradeoff, how risk should be measured must be addressed. Standard asset pricing theories such as the capital asset pricing model (Sharpe 1964) and the arbitrage pricing theory (Ross 1976) imply that expected returns should be positively related to systematic volatility, as measured through a factor model that summarises individual stock return exposure with respect to one or more rewarded risk factor(s). More recently, a series of papers has focused on the explanatory power of idiosyncratic, as opposed to systematic, risk for the cross-section of expected returns. Malkiel and Xu (2006), developing an insight from Merton (1987), show that an inability to hold the market portfolio, whatever the cause, will force investors to look, to some degree, at both total risk and market risk, so firms with larger firm-specific risk must deliver higher average returns to compensate investors for holding imperfectly diversified portfolios. That stocks with high idiosyncratic risk earn higher returns has also been confirmed in a number of recent empirical studies (Tinic and West 1986; Malkiel and Xu 1997, 2002).

Taken together, these findings suggest that total risk should be positively related to expected return. Most commonly, total risk is the volatility of a stock’s returns. Martellini (2008) has investigated the portfolio implications of these findings, and has found that tangency portfolios constructed on the assumption that the cross-section of excess expected returns could be approximated by the cross-section of volatility posted better out-of-sample risk-adjusted performance than their market-cap-weighted counterparts.

In this paper, we extend the results in Martellini (2008) to risk measures that take into account higher-order moments. Theoretical models have shown that, in exchange for higher skewness and lower kurtosis of returns, investors are willing to accept expected returns lower (and volatility higher) than those of the mean-variance benchmark (Rubinstein 1973; Kraus and Litzenberger 1976). More specifically,
skewness and kurtosis in individual stock returns (as opposed to the skewness and kurtosis of aggregate portfolios) have been shown to matter in several papers. High skewness is associated with lower expected returns in Barberis and Huang (2004), Brunnermeier, Gollier, and Parker (2007), and Mitton and Vorkink (2007). The intuition behind this result is that investors like to hold positively skewed portfolios. The highest skewness is achieved by concentrating portfolios in a small number of stocks that themselves have positively skewed returns. Thus investors tend to be underdiversified and drive up the price of stocks with high positive skewness, which in turn reduces their future expected returns. Stocks with negative skewness are relatively unattractive and thus have low prices and high returns. The preference for kurtosis is in the sense that investors like low kurtosis and thus expected returns should be positively related to kurtosis. Boyer, Mitton, and Vorkink (2009) and Conrad, Dittmar, and Ghysels (2008) provide empirical evidence that individual stocks’ skewness and kurtosis are indeed related to future returns.

An alternative to direct consideration of the higher moments of returns is to use a risk measure that aggregates the different dimensions of risk. In this line, Bali and Cakici (2004) show that future returns on stocks are positively related to their Value-at-Risk and Estrada (2000) and Chen, Chen, and Chen (2009) show that there is a relationship between downside risk and expected returns.

Our estimate of expected returns to construct the tangency portfolio proxy uses such a downside risk measure, and, in particular, the stock’s semi-deviation. The semi-deviation is a more meaningful definition of risk than volatility, since it takes into account only deviations below the mean. We compute the semi-deviation of the returns of each constituent $SEM_i$ with respect to the average return $\mu_i$ of the $i$-th stock as

$$SEM_i = \sqrt{\mathbb{E}\left\{\min(r_{i,t} - \mu_i, 0)\right\}^2}$$

where $\mathbb{E}(.)$ is the expectation operator computed as the arithmetic average, $\min(x,y)$ the minimum of $x$ and $y$, and $r_{i,t}$ the return of stock $i$ in week $t$.

To estimate expected returns, we follow the portfolio sorting approach of Fama and French (1992). That is, rather than attribute an expected return to each stock, we sort stocks by their total risk and form decile portfolios. We then attribute the median total risk of stocks in that decile portfolio to all stocks in the portfolio and use this risk measure as an estimate of expected return. The relationship between risk and returns derived from these portfolio sorts provides an estimate of expected returns.

Our expected return estimates are robust in the sense that they rely on a relation between return and risk that is firmly rooted in financial theory and that we refrain from estimating individual expected returns, instead sorting stocks into groups with high and low expected returns, consistent with cross-sectional asset pricing tests in the empirical finance literature.
2. Implementing Efficient Indexation
2. Implementing Efficient Indexation

We now turn to the implementation of portfolio optimisation with our robust input parameters, with the objective of deriving a reliable proxy for the tangency portfolio. This section describes the set of data used in our tests. In addition, practical implementation of the approach imposes further constraints, which we consider here. For example, our objective is to weight index constituents more efficiently, so we aim to match the actual constitution of the cap-weighted index as closely as possible. In addition, we introduce weight constraints and a method to control portfolio turnover.

To test our approach to constructing proxies for the tangency portfolio, we consider long-term US stock market data from CRSP. We consider the S&P 500 index and test whether we can improve its risk/return efficiency by weighting constituents differently than by their market capitalisation.

We obtain the S&P 500 constituent lists directly from CRSP, where one can see for each day which stocks belong to the index. For the risk-free rate, we use the ML US T-Bill 3M index from Datastream, and we compute the corresponding weekly returns. All equity returns time series are weekly total returns (including reinvestment of periodic payments such as dividends), as computed by CRSP. The constitution of the S&P 500 is available from 1959.

We assess portfolios, rebalanced every quarter, of all index constituents. The rebalancing is done after the close of the first Friday of January, April, July and October. To estimate optimal weights, we use returns for the two years before rebalancing. We select all constituents that are constituents of the underlying index at the rebalancing. To construct our index, we use the same constitution as that of the cap-weighted index.

We find the efficient weights as the set of weights that allow an investor to obtain the highest Sharpe ratio, given the risk and return inputs and the weight constraints. The constituent weights that solve this optimisation are the efficient weights $w^*$ that will be used in the efficient index, obtained with the following formula:

$$w^* = \arg \max_w \frac{w'\mu}{\sqrt{w'\Sigma w}}$$

where $\mu$ is the vector of expected returns in excess of the risk-free rate and $\Sigma$ is the covariance matrix for returns of these constituents. The efficient weights lead to the highest expected returns per unit of risk, given the expected excess returns and given the covariance matrix for the index constituents in question.

As described above, the covariance matrix is estimated from a statistical factor model using principal component analysis, whereas the expected returns are estimated through a risk/return relation, in which we sort stocks by total downside risk to group them into deciles according to their expected returns. Each quarter, we use the updated input parameters to derive optimal weights, implement these optimal weights at rebalancing and then hold the stocks until the next quarter.

We impose the usual portfolio constraint that weights have to sum to one. In addition, we impose weight constraints that depend on the number of constituents ($N$) in the index. We impose an upper bound
of $\lambda/N$ and a lower bound of $1/(\lambda M)$, where $\lambda$ is a flexibility parameter we set to two. Changing this parameter has no qualitative effect on the results. These constraints ensure that we include all index constituent stocks and that we do not obtain any negative weights that would lead to short sales. An appealing side effect of imposing weight constraints is that, not unlike statistical shrinkage techniques (Jagannathan and Ma 2003), it makes possible a better tradeoff between specification error and sampling error.

Although we wish to rebalance every quarter to be able to update information when necessary, it seems reasonable to rebalance the portfolio not at fixed intervals but only when weights have undergone significant shifts. This approach is consistent with insights from control techniques applied to portfolio optimisation to lower transaction costs (Leland 1999; El Bied, Martellini, and Priaulet 2002). To achieve lower turnover, we refrain from updating the optimal weights if the average absolute change in weights is less than 50% of the overall portfolio value. To ensure that we match the constitution of the underlying cap-weighted index, we continue to update the constitution in quarters in which we do not update the optimal weights. So exiting constituents will be deleted and new constituents will be included with the minimum weight $(1/\lambda N)$ at rebalancing. The table below shows the resulting turnover statistics and provides an analysis of indifference transaction costs. The table also shows results for another practical issue, portfolio concentration.

In addition to absolute turnover and to excess turnover relative to the cap-weighted index, exhibit 1 shows the impact this turnover would have on the performance of the efficient portfolios. Since transaction costs vary from one investor to another, it is unreasonable to assume fixed transaction costs. Instead, we compute the indifference level of transaction costs an investor would have to pay if these costs were to offset completely the difference in average returns compared to cap-weighting. This indifference level is 13% for the efficiently weighted portfolio. In practice, it is unlikely that any investor would pay costs of such magnitude. The improved risk/return efficiency of our weighting approach is thus robust to the occurrence of transaction costs.

Also of interest to investors is portfolio concentration. Indeed, it has been argued that one of the main drawbacks of

<table>
<thead>
<tr>
<th>Index</th>
<th>Annual one-way turnover</th>
<th>Excess turnover vs cap-weighted</th>
<th>Ann. return difference over cap-weighting</th>
<th>Indifference level of transaction costs</th>
<th>Average effective constituents</th>
<th>Effective constituents to nominal constituents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Efficient index</td>
<td>23.10%</td>
<td>18.41%</td>
<td>2.40%</td>
<td>13.06%</td>
<td>382</td>
<td>76%</td>
</tr>
<tr>
<td>Cap-weighted</td>
<td>4.69%</td>
<td>0.00%</td>
<td>-</td>
<td>-</td>
<td>94</td>
<td>19%</td>
</tr>
</tbody>
</table>

The table shows the resulting turnover measures for portfolios that have been implemented using controlled reoptimisation with a threshold value of 50%. The table also shows the indifference transaction costs, the difference between annualised return and cap-weighting divided by portfolio turnover. This measure indicates at which level (round-trip) transaction costs would neutralise the return difference with cap-weighting. The table also indicates the effective number of constituents in the efficient index and in the cap-weighted index, computed as the inverse of the sum of squared constituent weights. This measure is computed at the start of each quarter and averaged over the entire period. The results are based on weekly return data from 01/1959 to 12/2008 for S&P 500 constituents.
capitalisation weighting is excessive concentration in a few stocks with high market capitalisation. The argument is that since few stocks will account for most of the weight in the index the effective number of stocks held in a cap-weighted index will be well below the actual number of constituents. We follow Strongin, Petsch, and Sharenow (2000) in computing the effective number of stocks as the inverse of the sum of squared portfolio weights. For the S&P 500 universe, the efficient weighting method leads to portfolios with an average of 382 effective constituents, whereas the cap-weighted index has only 94 stocks effectively by this measure. Thus, with efficiently weighted portfolios concentration is considerably reduced.

2. Implementing Efficient Indexation
3. Performance of Efficient Indexation
Now that we have described a method that controls turnover and shown the feasibility of the approach in terms of portfolio turnover and concentration, we turn to an analysis of the risk and return properties of the strategy. As our approach is an alternative to cap-weighted stock market indices that is based on the exact same constituents and changes only the weighting scheme, risk and return statistics for the cap-weighted index are shown for comparison. As the efficient index and the cap-weighted index have exactly the same constituents, the resulting portfolios will show commonalities in risk and return. At an annualised 5%, the tracking error of efficient indexation is lower than that of the cap-weighted index. This section looks first at long-term performance and then at the consistency of performance across market conditions.

3.1. Long-Term Risk and Return
Both the absolute and the relative performance of the strategy must be analysed. In addition, it is necessary to test whether the efficient weighting method’s outperformance of capitalisation weighting is statistically significant and to assess exposure to extreme risks and volatility. This section does these analyses for the full historical time period, whereas the next section will focus on performance in different market conditions.

Exhibit 2 shows summary performance statistics. For the average return, volatility and the Sharpe ratio, we report differences with respect to cap-weighting and assess whether this difference is statistically significant. It is important to assess significance, as we base our conclusions on a limited amount of data, and any differences could, in principle, be the result of random effects.

Exhibit 2 shows that the efficient weighting of index constituents leads to higher average returns, lower volatility, and a higher Sharpe ratio. All these differences are statistically significant at the 10% level, whereas the difference in Sharpe ratios is significant even at the 0.1% level. Given the data, it is highly unlikely that the unobservable true performance of efficient weighting was not different from that of capitalisation weighting. Economically, the performance difference is pronounced, as the Sharpe ratio increases by about 70%.

Exhibit 2: Risk and return

<table>
<thead>
<tr>
<th>Index</th>
<th>Ann. average return (compounded)</th>
<th>Ann. standard deviation</th>
<th>Sharpe ratio (compounded)</th>
<th>Information ratio</th>
<th>Tracking error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Efficient index</td>
<td>11.63%</td>
<td>14.65%</td>
<td>0.41</td>
<td>0.52</td>
<td>4.65%</td>
</tr>
<tr>
<td>Cap-weighted</td>
<td>9.23%</td>
<td>15.20%</td>
<td>0.24</td>
<td>0.00</td>
<td>0.00%</td>
</tr>
<tr>
<td>Difference (efficient minus cap-weighted)</td>
<td>2.40%</td>
<td>-0.55%</td>
<td>0.17</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>p-value for difference</td>
<td>0.14%</td>
<td>6.04%</td>
<td>0.04%</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

The table shows risk and return statistics portfolios constructed with the same set of constituents as the cap-weighted index. Rebalancing is quarterly subject to an optimal control of portfolio turnover (by setting the reoptimisation threshold to 50%). Portfolios are constructed by maximising the Sharpe ratio given an expected return estimate and a covariance estimate. The expected return estimate is set to the median total risk of stocks in the same decile when sorting by total risk. The covariance matrix is estimated using an implicit factor model for stock returns. Weight constraints are set so that each stock’s weight is between $1/2N$ and $2/N$, where $N$ is the number of index constituents. P-values for differences are computed using the paired t-test for the average, the F-test for volatility, and a Jobson-Korkie test for the Sharpe ratio. The results are based on weekly return data from 01/1959 to 12/2008.
3. Performance of Efficient Indexation

The performance measures used above adjust portfolio returns for absolute risk, i.e., for the variability in portfolio wealth without reference to an external benchmark. Since the efficient weighting procedure is an alternative to cap-weighted indexing for investors seeking exposure to the risk premium in equity markets, the standard cap-weighted index is a useful benchmark. We measure the performance of our index relative to the cap-weighted benchmark by computing alpha and beta from a single-factor analysis. This corresponds to a CAPM framework, in which the cap-weighted index is taken as a proxy for the market portfolio. Exhibit 3 shows the performance of the efficient indexation method once we adjust for its exposure to market risk in the sense of its beta with the cap-weighted index.

In spite of the favourable absolute and relative performance of the efficient indexation method, it is interesting to analyse if the strategy exposes investors to other forms of risk. In particular, our analysis has focused on measures that do not take into account the presence of extreme risks. We ask whether the greater risk/reward efficiency in terms of the volatility of these indices comes at the cost of a higher risk of extreme losses. We first compute aggregate measures of extreme or downside risk, notably Value-at-Risk and semi-deviation. We compute Value-at-Risk to estimate the worst loss an investor can expect to incur over a weekly horizon with 95% confidence. Our Value-at-Risk estimate follows Zangari (1996) and takes into account not only the volatility but also the skewness and kurtosis of the return distribution. Portfolio semi-deviation is computed much as is the individual stock's semi-deviation.

The results in the table show that for the S&P universe the efficient indexation method significantly outperforms the cap-weighted benchmarks, since the intercept of the regression is significant. The annualised alpha is on the same order of magnitude as the annualised return difference in exhibit 2, suggesting that the higher returns of the efficient indexation strategy are not caused by greater exposure to market risk.

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**Exhibit 3: CAPM analysis**

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Ann. alpha</th>
<th>Beta</th>
<th>R-squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.92%</td>
<td>91.77%</td>
<td>91%</td>
<td></td>
</tr>
<tr>
<td>t-statistic</td>
<td>3.7</td>
<td>68.3</td>
<td></td>
</tr>
<tr>
<td>p-value</td>
<td>0.02%</td>
<td>0.00%</td>
<td></td>
</tr>
</tbody>
</table>

The table shows coefficient estimates from a regression of weekly returns of the efficient indexation strategy on weekly returns of the cap-weighted index. P-values are obtained using Newey-West robust standards that are consistent in the case of heteroscedasticity and autocorrelation. The data are for the period from 01/1959 to 12/2008.

**Exhibit 4: Extreme risk**

<table>
<thead>
<tr>
<th>Index</th>
<th>95% Value-at-Risk over one week</th>
<th>Ann. semi-deviation</th>
<th>3-month trailing return</th>
<th>12-month trailing return</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1st percentile</td>
<td>5th percentile</td>
</tr>
<tr>
<td>Efficient Index</td>
<td>3.20%</td>
<td>10.93%</td>
<td>-21.89%</td>
<td>-10.72%</td>
</tr>
<tr>
<td>Cap-weighted Index</td>
<td>3.28%</td>
<td>11.13%</td>
<td>-20.99%</td>
<td>-10.12%</td>
</tr>
</tbody>
</table>

The table shows different measures of downside risk for efficient indexation and cap-weighting. The 95% Value-at-Risk is computed using a Cornish-Fisher expansion. Semi-deviation is the square-root of the second lower partial moment with respect to the mean and annualised by multiplying with the square-root of 52. Three-month trailing returns are computed by compounding the past thirteen weeks of returns for each weekly observation, and twelve-month trailing returns by compounding the past fifty-two weeks of returns. The table shows percentiles for the distribution of the available sample of trailing returns. The results are based on weekly return data from 01/1959 to 12/2008.
3. Performance of Efficient Indexation

The VaR and semi-deviation for the efficient index are lower than those of the cap-weighted index. We can thus conclude that the improvement in the volatility-adjusted return (the higher Sharpe ratio) does not come at the cost of higher downside risk. This result suggests that, though we tend to overweight stocks with high downside risk through the expected return estimate, this risk is diversified away on the portfolio level. We also look at more ad hoc measures of downside risk by computing short-term trailing returns. We look at three- and twelve-month horizons. To assess the loss risk for an investor, we look at the most negative trailing returns. The table above shows the first, fifth, and tenth percentiles, i.e., the rolling return that is exceeded in 99%, 95%, and 90% of the sample. We can see that twelve-month trailing losses are considerably lower for the efficient index than for the cap-weighted index. Three-month trailing losses are broadly similar to those of the cap-weighted index. If anything, the extreme risk inherent to the efficient weighting approach is lower than that of cap-weighted indices.

3.2. Efficient Indexation versus Naïve De-concentration

Exhibit 1 shows that the efficient indexation strategy leads to a portfolio that is considerably less concentrated than its cap-weighted counterpart. A different way to limit concentration would simply be to weight each stock equally. This naïve form of de-concentration ignores any possibility of portfolio optimisation. It seems useful to compare the performance of the efficient indexation strategy and that of this naïve alternative. In fact, if the performance of efficient indexation could be attributed to a mere de-concentration effect, we would expect the performance of the equal-weighted strategy to be even stronger than that of the efficient indexation strategy, as concentration is, by definition, lower for an equal-weighted portfolio.

Exhibit 5 shows that efficient indexation based on robust portfolio construction seems preferable to a simple equal-weighting scheme. This suggests that portfolio optimisation with robust parameter estimates, as introduced in section 1, adds more useful information than an equal-weighted benchmark. The table below shows, in particular, that efficient indexation leads to higher expected returns and lower volatility than its equal-weighted counterpart. The tracking error and turnover of the efficient indexation strategy are also slightly lower than those of the equal-weighted strategy.

The bottom line from exhibit 5 is that efficient indexation leads to Sharpe and information ratios considerably higher than does equal weighting. That efficient

<table>
<thead>
<tr>
<th></th>
<th>Ann. average return</th>
<th>Ann. standard deviation</th>
<th>Sharpe ratio</th>
<th>Info ratio</th>
<th>Tracking error</th>
<th>Annual one-way turnover</th>
<th>Effective constituents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equal weighting</td>
<td>11.1%</td>
<td>15.8%</td>
<td>0.35</td>
<td>0.39</td>
<td>4.8%</td>
<td>24.2%</td>
<td>500</td>
</tr>
<tr>
<td>Efficient index</td>
<td>11.6%</td>
<td>14.6%</td>
<td>0.41</td>
<td>0.52</td>
<td>4.7%</td>
<td>23.1%</td>
<td>382</td>
</tr>
</tbody>
</table>

The table shows risk and return statistics portfolios constructed with the same set of constituents as the cap-weighted index. The efficient indexation method from above is compared to the equal-weighted portfolio with quarterly rebalancing that is based on the same set of constituents. The results are based on weekly return data from 01/1959 to 12/2008.
3. Performance of Efficient Indexation

Indexation makes possible performance superior to equal weighting, and with a lower effective number of constituents, also suggests that the efficient indexation method is suitable for constituent universes that include stocks that have low liquidity. As the effective number of stocks of the efficient indexation strategy is relatively low, it is possible to avoid holding the least liquid stocks. In practice, then, transaction and liquidity costs may be lower for efficient indexation.

3.3. A Closer Look at the Performance of Efficient Indexation

The evidence provided above suggests that the efficient indexation method greatly improves the risk/return efficiency of cap-weighted indices. In fact, Sharpe ratios are considerably higher than those of cap-weighted indices, even though the underlying constituents are identical. The analysis above is based on long-term historical data. For the investor, it is important that the improvement in risk/reward efficiency be consistent. To determine whether it is, we provide an overview of how efficient indexation fares in different time periods, stock market regimes, and economic conditions.

The upper graph in exhibit 6 shows the growth over time of investments in the efficient index and in the cap-weighted index. The plots for the two constituent universes show that the return difference leads to spectacular differences in wealth over long time periods, the simple result of compounding.

For an idea about the consistency of the increase in returns through efficient indexation, we also plot the ratio of the portfolio wealth obtained with efficient indexation to the wealth obtained by cap-weighting the same stocks. The lower plot shows the ratio of the wealth of an investor in the efficient index to the wealth of an investor in the value-weighted index at each point in time, assuming that both investors start investing at the same date and with the same amount. Thus the plot shows how many dollars an investor in the efficient index has for every dollar he would have had when investing in the value-weighted index.

Exhibit 6: Growth of portfolio wealth

The upper graph shows cumulative returns normalised to a starting value of one for efficient indexation and for cap-weighting. The lower graph indicates the ratio of the solid line to the dotted line in the upper plot. The results are for the efficient indexation portfolios with controlled reoptimisation. The data have a weekly frequency and range from 01/1959 to 12/2008.

The graph shows that, over time, the efficient index’s cumulative outperformance of the cap-weighted index is considerable. Efficient indexation does, however, underperform value weighting in the years from January 1996 to December 1999. Wealth ratios for both indices fall over this period, the time of the bull markets that led to the “tech
3. Performance of Efficient Indexation

bubble”. Except for this period, the wealth ratio either increases or is stable, suggesting that the method provides a consistent return enhancement other than in the period of the extremely bullish markets of the late 1990s.

The long-term evolution of wealth highlights average returns rather than risk and risk-adjusted performance. We analyse performance statistics over sub-samples for a more systematic picture of the variations in performance by time period. Exhibit 7 shows the annualised return, volatility, and Sharpe ratio for periods of a decade. We divided the sample into non-overlapping periods of ten years, going backwards from December 2008. We thus obtain five sub-periods of ten years.

Exhibit 7 shows that the Sharpe ratio of efficient indexation is higher in every ten-year period but that from 1989 to 1998. This confirms the underperformance in bull markets observed in the graphs on the growth of wealth. Interestingly, the underperformance in bull markets suggests that the performance of efficient indexation is, in general, more stable than that of cap-weighting. In addition, though the Sharpe ratio of efficient indexation is lower than that of the cap-weighted index in the bull markets of the 1990s, efficient indexation is less volatile over this period.

It is useful to look directly at the dependence of performance on cap-weighted market returns. We take the non-parametric regression approach of Fung and Hsieh (1997) to assess the dependence of returns on the market factor. We sort all weekly observations by the returns of the cap-weighted index and form quintiles. The first quintile contains the 20% of weeks in the sample that have the lowest returns for the cap-weighted index. The fifth quintile contains the 20% of the weeks in the sample that have the highest returns. Computing the average weekly returns for each quintile shows how the strategy depends on the returns of the cap-weighted index.

Exhibit 7: Risk and return in different decades

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1999-2008</td>
<td>-1.22%</td>
<td>3.47%</td>
<td>18.98%</td>
<td>18.04%</td>
<td>-0.23</td>
<td>0.01</td>
</tr>
<tr>
<td>1989-1998</td>
<td>19.16%</td>
<td>16.43%</td>
<td>12.84%</td>
<td>12.45%</td>
<td>1.07</td>
<td>0.89</td>
</tr>
<tr>
<td>1979-1988</td>
<td>16.32%</td>
<td>20.82%</td>
<td>16.02%</td>
<td>15.82%</td>
<td>0.42</td>
<td>0.71</td>
</tr>
<tr>
<td>1969-1978</td>
<td>2.96%</td>
<td>4.24%</td>
<td>16.02%</td>
<td>15.47%</td>
<td>-0.20</td>
<td>-0.13</td>
</tr>
<tr>
<td>1959-1968</td>
<td>10.33%</td>
<td>14.29%</td>
<td>10.65%</td>
<td>10.05%</td>
<td>0.62</td>
<td>1.05</td>
</tr>
</tbody>
</table>

The table shows risk and return statistics when dividing the sample into periods of ten years. The results are based on weekly return data from 01/1959 to 12/2008.

Exhibit 8: Dependence of returns on cap-weighted returns

<table>
<thead>
<tr>
<th>Quintile (by cap-weighted return)</th>
<th>Average weekly return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Cap-weighted -2.68%</td>
</tr>
<tr>
<td>2</td>
<td>Cap-weighted -0.71%</td>
</tr>
<tr>
<td>3</td>
<td>Cap-weighted 0.31%</td>
</tr>
<tr>
<td>4</td>
<td>Cap-weighted 1.16%</td>
</tr>
<tr>
<td>5</td>
<td>Cap-weighted 2.89%</td>
</tr>
<tr>
<td>Efficient indexation</td>
<td>Cap-weighted -2.46%</td>
</tr>
<tr>
<td></td>
<td>Efficient indexation -0.57%</td>
</tr>
<tr>
<td></td>
<td>Efficient indexation 0.39%</td>
</tr>
<tr>
<td></td>
<td>Efficient indexation 1.16%</td>
</tr>
<tr>
<td></td>
<td>Efficient indexation 2.63%</td>
</tr>
</tbody>
</table>

The table shows average returns computed for five sub-samples of equal size. The sub-samples are obtained by sorting the weekly observations based on the weekly return of the cap-weighted index. The results are based on weekly return data from 01/1959 to 12/2008.
3. Performance of Efficient Indexation

Exhibit 8 shows that efficient indexation has higher average returns than cap-weighting in all quintiles except the highest. The results in exhibit 8 confirm, unsurprisingly, that efficient indexation has lower returns than cap-weighting bull markets. An intuitive explanation is that it is extremely difficult to outperform the trend-following strategy when markets continue to follow the trend and the stocks with price increases continue to go up. However, the dispersion of efficient-weighted portfolio returns across quintiles is also lower, again suggesting more stability.

Conditioning on the cap-weighted return does not provide a complete characterisation of varying market conditions. For a look at economic conditions in a broader sense, we repeat the analysis of exhibit 7, in which we divided the sample into sub-samples, and computed performance statistics. This time, we sort the sample into sub-samples according to the prevailing economic conditions. To characterise economic conditions, we use two variables. The first is a recession indicator for the US economy, which we obtain from the NBER. The second is implied volatility, computed by the CBOE based on option prices for index options written on the S&P index.

Exhibit 9 shows results separately for recessionary and expansionary periods.

The results show that both capitalisation weighting and efficient indexation fare much better in expansions than in recessions. In recessions, average returns are lower and volatility of returns is higher. In both stages of the business cycle, efficient indexation provides higher average returns, lower volatility, and thus higher Sharpe ratios.

Another useful conditioning variable is implied volatility. Although the recession variable used above tells us something about the realisation of economic variables, option-implied volatility directly captures investor uncertainty. The advantage of using option-implied volatility rather than realised volatility is that we can measure implied volatility precisely at a weekly frequency. In addition, implied volatility, which Whaley (2000) has described as a “fear gauge”, directly reflects investor’s instantaneous beliefs and preferences rather than past realisations. Exhibit 10 repeats the analysis from the previous table. The difference is that the sub-samples are now formed according to volatility regimes. Data on implied volatility indices are available only from 1986 for the S&P index. We divide the data available since 1986 into one half that corresponds to high volatility weeks and another half to low volatility weeks.

<table>
<thead>
<tr>
<th>&quot;Business cycle&quot;</th>
<th>Ann. average return</th>
<th>Ann. volatility</th>
<th>Sharpe ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cap-weighting</td>
<td>Efficient indexation</td>
<td>Cap-weighting</td>
</tr>
<tr>
<td>Recessions</td>
<td>-1.64%</td>
<td>2.26%</td>
<td>22.85%</td>
</tr>
<tr>
<td>Expansions</td>
<td>11.19%</td>
<td>13.30%</td>
<td>13.47%</td>
</tr>
</tbody>
</table>

The table shows risk and return statistics computed for two sub-samples. The sub-samples are obtained by sorting the weekly observations based on a recession indicator for that week. The recession indicator is obtained from NBER dates for peaks and troughs of the business cycle. The results are based on weekly return data from 01/1959 to 12/2008.
As it does in both recessions and periods of growth, efficient indexation improves risk/return efficiency in both times of great uncertainty and times of low uncertainty. Its advantage over capitalisation weighting in terms of reduced volatility is most pronounced in times of great uncertainty, suggesting that efficiently weighted portfolios provide risk reduction benefits precisely when they are most needed.

In general, when the performance of our indexing method conditional on time, market conditions or economic conditions is analysed, the improvements in risk/reward efficiency are confirmed. In fact, the performance of efficient indexation is extremely robust, regardless of the time period, point on the business cycle, or degree of uncertainty. When returns in rising and falling markets are analysed, we find that efficient indexation lags capitalisation weighting in pronounced bull markets, as in the late 1990s. From an investor’s perspective, however, underperforming capitalisation weighting when it returns 20% or more per year may be a risk worth taking in exchange for greater average efficiency.

Exhibit 10: Risk and return in times of high uncertainty and low uncertainty

<table>
<thead>
<tr>
<th>&quot;Implied volatility regime&quot;</th>
<th>Ann. average return</th>
<th>Ann. volatility</th>
<th>Sharpe ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cap-weighting</td>
<td>Efficient indexation</td>
<td>Cap-weighting</td>
</tr>
<tr>
<td>High volatility</td>
<td>8.90%</td>
<td>10.99%</td>
<td>15.40%</td>
</tr>
<tr>
<td>Low volatility</td>
<td>6.22%</td>
<td>10.03%</td>
<td>11.88%</td>
</tr>
</tbody>
</table>

The table shows risk and return statistics computed for two sub-samples of equal size. The sub-samples are obtained by sorting the weekly observations based on the value of the corresponding implied volatility index for that week. The median level of volatility is used to separate the two samples. The data for implied volatility indices start on 03/01/1986 (VXO index) and end on 26/12/2008.
4. International Evidence
4. International Evidence

Although the results obtained for post-war US data suggest that the improvement in efficiency is highly significant both statistically and economically, it may be that these results are specific to US data. So it is important to gather evidence on how efficient indexation fares internationally. Since it is more challenging to obtain accurate data over long time periods for international markets, we analyse indices only for countries or regions with the largest stock market capitalisations and we concentrate on a recent time period for which data are available.

We apply the efficient indexation method with the same parameters as above to the constituents of the FTSE All World index from the following countries or regions: USA, Eurobloc, United Kingdom, developed Asia-Pacific ex Japan (including stocks from Australia, Hong Kong, New Zealand, and Singapore), and Japan. For these indices, we obtain daily constituent lists and constituent-level return data for a period of approximately seven years (from 23/12/2002 to 18/09/2009). Exhibit 11 shows the risk and return statistics obtained through efficient indexation based on these constituents and compares them to the corresponding statistics of the FTSE All World indices that weight constituent stocks by (free-float-adjusted) market capitalisation.

The results in exhibit 11 show that risk/return efficiency in terms of the Sharpe ratio is improved considerably for all five indices. In addition, the improvement is actually quite similar across the five indices, with Sharpe ratios approximately 0.15 higher than those of the cap-weighted index. When the results for the four international indices and for the US index are compared, it is clear that the method works slightly better in the other datasets, except perhaps in the Asia Pacific index. It is interesting to note that the Asia Pacific index had extremely high returns of more than 15% over the period, compared to returns of less than 6% for all other cap-weighted indices. Thus, the lesser improvement of the Sharpe ratio through efficient indexation in this dataset is actually coherent with the observation in the sub-sample analysis for the long-term US data, where it was found that, in strong bull markets, efficient indexation does not improve on capitalisation weighting as much as it does in other market conditions.

In general, analysis of international data suggests that our results are not specific to US data, as the method yields similar results in stock markets around the world.

<table>
<thead>
<tr>
<th>Index</th>
<th>Ann. average return</th>
<th>Ann. std. dev</th>
<th>Sharpe ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>USA</td>
<td>5.60%</td>
<td>2.77%</td>
<td>2.83%</td>
</tr>
<tr>
<td>Eurobloc</td>
<td>7.48%</td>
<td>4.19%</td>
<td>3.30%</td>
</tr>
<tr>
<td>UK</td>
<td>9.66%</td>
<td>5.44%</td>
<td>4.23%</td>
</tr>
<tr>
<td>Asia</td>
<td>17.19%</td>
<td>15.80%</td>
<td>1.40%</td>
</tr>
<tr>
<td>Japan</td>
<td>5.85%</td>
<td>3.01%</td>
<td>2.84%</td>
</tr>
</tbody>
</table>

The table shows risk and return statistics computed for efficient indexation and capitalisation weighting applied to stock market index constituents in five regions. The statistics are based on daily returns data from 23/12/2002 to 18/09/2009.
Conclusion
Evidence abounds of the inefficiency of cap-weighted indices. Currently available alternatives may well owe their success to that inefficiency, but, surprisingly, they do not explicitly address this problem. Characteristics-based indices, for example, attempt to be more representative of the economy by weighting stocks by each company’s economic footprint. Their goal is not to weight stocks so as to improve risk/return efficiency. The approach described here, on the other hand, takes the investor’s perspective into account and makes risk/return efficiency an explicit goal. Input parameters throw up a major conceptual obstacle to constructing efficient portfolios, as estimation error may weaken optimisation results. From a practical standpoint, optimisation methods may lead to high turnover and thus to transaction costs that wipe out favourable performance. In this paper, we draw on the academic literature to provide solutions to both the parameter estimation problem and the turnover problem. Our main contribution is to provide a novel approach, focusing on efficiency, to equity indexation; efficiency, after all, was arguably the motivation for the creation of index funds drawing on insights from the CAPM in the first place.

Our implementation of the fundamental insight of modern portfolio theory, that investors should hold the tangency portfolio, is based on robust estimates of risk and return parameters. To obtain robust parameter estimates for the comovements of stock returns, we use a multifactor model based on principal component analysis. For expected returns, we use the insight that there is a risk/return tradeoff and generate estimates from a suitably designed risk measure that involves not only average risk but also higher moment risk, following the evidence of the link between downside risk and expected returns provided by Estrada (2000) and Chen, Chen, and Chen (2009), as well as the evidence of the importance of total risk for portfolio construction in Martellini (2008). Out of practical concerns, we also introduce a procedure, inspired by optimal control theory (Leland 1999), to control turnover and transaction costs.

The empirical tests described in this paper show that this procedure allows us to generate efficient indices with out-of-sample Sharpe ratios considerably higher than those of their capitalisation-weighted counterparts. In addition, performance is consistent across different business cycles, volatility regimes, and time periods. Lower risk/return efficiency occurs only in the extreme bull markets of the late 1990s. Even in this period, efficient indexation posted lower volatility than capitalisation weighting, and expected returns were lower when the cap-weighted indices were returning in excess of 20% a year. It should also be kept in mind that, unlike that of other index construction methods that do not weight constituents by market capitalisation, the performance of the method can be put down entirely to a different method of weighting constituents.

On the whole, when the evidence from post-war US data is taken into account, the differences in the efficiency of value-weighted indexation and efficient indexation (and efficient indexation is more efficient) are statistically significant. The increase in risk/return efficiency is similar when the method is applied to
international stock market indices. In general, efficient indexation leads to an economically significant increase in efficiency for investors seeking exposure to the equity risk premium.
Conclusion
References
References


References


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About EDHEC-Risk Institute
About EDHEC-Risk Institute

The Choice of Asset Allocation and Risk Management

EDHEC-Risk structures all of its research work around asset allocation and risk management. This issue corresponds to a genuine expectation from the market. On the one hand, the prevailing stock market situation in recent years has shown the limitations of diversification alone as a risk management technique and the usefulness of approaches based on dynamic portfolio allocation. On the other, the appearance of new asset classes (hedge funds, private equity, real assets), with risk profiles that are very different from those of the traditional investment universe, constitutes a new opportunity and challenge for the implementation of allocation in an asset management or asset-liability management context. This strategic choice is applied to all of the centre’s research programmes, whether they involve proposing new methods of strategic allocation, which integrate the alternative class; taking extreme risks into account in portfolio construction; studying the usefulness of derivatives in implementing asset-liability management approaches; or orienting the concept of dynamic “core-satellite” investment management in the framework of absolute return or target-date funds.

An Applied Research Approach

In an attempt to ensure that the research it carries out is truly applicable, EDHEC has implemented a dual validation system for the work of EDHEC-Risk. All research work must be part of a research programme, the relevance and goals of which have been validated from both an academic and a business viewpoint by the centre’s advisory board. This board is made up of internationally recognised researchers, the centre’s business partners and representatives of major international institutional investors. The management of the research programmes respects a rigorous validation process, which guarantees the scientific quality and the operational usefulness of the programmes.

Six research programmes have been conducted by the centre to date:

- Asset allocation and alternative diversification
- Style and performance analysis
- Indices and benchmarking
- Operational risks and performance
- Asset allocation and derivative instruments
- ALM and asset management

These programmes receive the support of a large number of financial companies. The results of the research programmes are disseminated through the three EDHEC-Risk locations in London, Nice and Singapore.

In addition, EDHEC-Risk has developed a close partnership with a small number of sponsors within the framework of research chairs. These research chairs correspond to a commitment over three years from the partner on research themes that are agreed in common.
About EDHEC-Risk Institute

The following research chairs have been endowed to date:

- Regulation and Institutional Investment, in partnership with AXA Investment Managers (AXA IM)
- Asset-Liability Management and Institutional Investment Management, in partnership with BNP Paribas Investment Partners
- Risk and Regulation in the European Fund Management Industry, in partnership with CACEIS
- Structured Products and Derivative Instruments, sponsored by the French Banking Federation (FBF)
- Private Asset-Liability Management, in partnership with ORTEC Finance
- Dynamic Allocation Models and New Forms of Target-Date Funds, in partnership with UFG
- Advanced Modelling for Alternative Investments, in partnership with Newedge Prime Brokerage
- Asset-Liability Management Techniques for Sovereign Wealth Fund Management, in partnership with Deutsche Bank
- Core-Satellite and ETF Investment, in partnership with Amundi Investment Solutions
- The Case for Inflation-Linked Bonds: Issuers’ and Investors’ Perspectives, in partnership with Rothschild & Cie

Each year, EDHEC-Risk organises a major international conference for institutional investors and investment management professionals with a view to presenting the results of its research: the EDHEC-Risk Institutional Days.

EDHEC also provides professionals with access to its website, www.edhec-risk.com, which is entirely devoted to international asset management research. The website, which has more than 35,000 regular visitors, is aimed at professionals who wish to benefit from EDHEC's analysis and expertise in the area of applied portfolio management research. Its monthly newsletter is distributed to more than 400,000 readers.

### EDHEC-Risk Institute: Key Figures, 2008–2009

<table>
<thead>
<tr>
<th>Category</th>
<th>Figures</th>
</tr>
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<tbody>
<tr>
<td>Number of permanent staff</td>
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<td>Number of research associates</td>
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<td>Number of affiliate professors</td>
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<tr>
<td>Number of participants at EDHEC-Risk Executive Education seminars</td>
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</tr>
</tbody>
</table>

### Research for Business

The centre’s activities have also given rise to executive education and research service offshoots.

EDHEC-Risk’s executive education programmes help investment professionals to upgrade their skills with advanced risk and asset management training across traditional and alternative classes.
About EDHEC-Risk Institute

The EDHEC-Risk Institute PhD in Finance
The EDHEC-Risk Institute PhD in Finance at EDHEC Business School is designed for professionals who aspire to higher intellectual levels and aim to redefine the investment banking and asset management industries. It is offered in two tracks: a residential track for high-potential graduate students, who hold part-time positions at EDHEC Business School, and an executive track for practitioners who keep their full-time jobs. Drawing its faculty from the world's best universities and enjoying the support of the research centre with the greatest impact on the European financial industry, the EDHEC-Risk Institute PhD in Finance creates an extraordinary platform for professional development and industry innovation.

The EDHEC-Risk Institute MSc in Risk and Investment Management
The EDHEC-Risk Institute Executive MSc in Risk and Investment Management is designed for professionals in the investment management industry who wish to progress, or maintain leadership in their field, and for other finance practitioners who are contemplating lateral moves. It appeals to senior executives, investment and risk managers or advisors, and analysts. This postgraduate programme is designed to be completed in seventeen months of part-time study and is formatted to be compatible with professional schedules.

The programme has two tracks: an executive track for practitioners with significant investment management experience and an apprenticeship track for selected high-potential graduate students who have recently joined the industry. The programme is offered in Asia—from Singapore—and in Europe—from London and Nice.

FTSE EDHEC-Risk Efficient Indices
FTSE Group, the award-winning global index provider, and EDHEC-Risk Institute launched the first set of FTSE EDHEC-Risk Efficient Indices in early 2010. Initially offered for the UK, the Eurobloc, the USA, Developed Asia-Pacific ex-Japan, and Japan, the index series aims to capture equity market returns with an improved risk/reward efficiency compared to cap-weighted indices. The weighting of the portfolio of constituents achieves the highest possible return-to-risk efficiency by maximising the Sharpe ratio (the reward of an investment per unit of risk).

EDHEC-Risk Alternative Indexes
The different hedge fund indexes available on the market are computed from different data, according to diverse fund selection criteria and index construction methods; they unsurprisingly tell very different stories. Challenged by this heterogeneity, investors cannot rely on competing hedge fund indexes to obtain a “true and fair” view of performance and are at a loss when selecting benchmarks. To address this issue, EDHEC-Risk was the first to launch composite hedge fund strategy indexes as early as 2003.

The thirteen EDHEC-Risk Alternative Indexes are published monthly on www.edhec-risk.com and are freely available to managers and investors.
EDHEC-Risk Institute
EDHEC-Risk Institute Publications (2007-2010)

2010
- Goltz, F., and Le Sourd, V. Does finance theory make the case for capitalisation-weighted indexing? (January)

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