Proverbial Baskets are Uncorrelated Risk Factors!
A Factor-Based Framework for Measuring and Managing Diversification in Multi-Asset Investment Solutions

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The investment industry has historically focused on security selection decisions—a focus that has come as a distraction with respect to another key source of added value, namely, asset allocation decisions. In the face of recent crises, and given the intrinsic difficulty of delivering added value through security selection decisions alone, the relevance of the old paradigm has been questioned with heightened intensity, and the focus is shifting toward the proper management of factor exposures as the main source of performance. In this broader context, multi-asset investment solutions have become increasingly popular among sophisticated institutional investors focusing on more efficient approaches for harvesting risk premia across and within asset classes.

The key insight from portfolio theory is that *diversification*, as opposed to *hedging* or *insurance*, is the risk management technique that should be used to efficiently eliminate unrewarded risk exposures from investors' portfolios, allowing them to earn the highest level of expected return for a given risk budget. According to this definition, well-diversified portfolios are scientifically defined as portfolios that deliver the highest reward per unit of risk. Taking volatility as the risk measure, this leaves us with a simple prescription—namely, maximize the Sharpe ratio. This definition, however, is not fully operational because of the presence of parameter uncertainty, in particular in expected returns [Merton 1980]; and in practice, proxies for the maximum Sharpe ratio (MSR) portfolio do not necessarily have a consistently higher out-of-sample Sharpe ratio than a portfolio that simply allocates the same proportion of wealth to each asset [DeMiguel et al. [2009].

In view of this problem, managers of multi-asset portfolios may be tempted to go back to a more heuristic approach to diversification, summarized in the principle that "eggs should be spread across many baskets." Of course, this prescription is too vague to translate into an operational asset allocation policy until a proper meaning is given to "eggs" and "baskets." The equally weighted portfolio spreads dollars evenly across assets, but it disregards any information about cross-sectional difference in riskiness. As a result, the risk of an equally weighted portfolio is often concentrated in the most volatile constituent (Qian [2005] and Maillard et al. [2010]).

The *risk parity* interpretation for the eggs-and-baskets principle is to consider that eggs are risk contributions, as opposed to dollar contributions, and that baskets are assets, so that the highest level of naive diversification is achieved by the "equal risk contribution" portfolio, as opposed to the "equal dollar allocation" portfolio.

On the other hand, diversification in multi-asset strategies can still be illusory if all constituents contribute equally to risk but are highly correlated, at least conditionally upon extreme equity market downturns, and essentially load on the same risk factor. This lesson, which was painfully experienced in 2008 by a large number of endowments and other investors holding seemingly well-diversified multi-asset portfolios, suggests that if eggs should rightfully be regarded as risk contributions, baskets should refer to uncorrelated underlying factors, as opposed to correlated asset classes.

Academic research has provided some normative background for the factor perspective. Ang, Goetzmann, and Schaefer [2009] show that a factor model helps understand the disappointing active returns of the Norwegian sovereign fund in 2008 and 2009. More generally, given that security and asset class returns can be explained by their exposure to pervasive systematic risk factors, looking through asset class decompositions to focus on underlying factor decompositions appears to be a perfectly legitimate approach, which is formally supported by the arbitrage pricing theory of Ross [1976]. The broad objective pursued by adopting a factor perspective is to understand the underlying sources of risk and return of various assets so as to better assess the level of diversification in a multi-asset portfolio. This is eloquently illustrated by Ang [2014, p. 194]: "Factors are to assets what nutrients are to food." In the same way that different types of food may in fact contain identical nutrients, leaving many of us with a seemingly balanced diet that is effectively excessively biased in favor of carbohydrates, factor exposures can overlap across different assets to form a multi-asset portfolio excessively biased in favor of a single factor exposure.

The main objective of this article is to show that a factor-based investment approach can allow for a better structuration of the investment process in the design of multi-asset investment solutions. Our analysis builds upon the concept of *effective number of bets* (ENB), introduced by Meucci [2009b], which is formally defined as a measure of dispersion of the distribution of factor contributions to portfolio volatility. We present an empirical application of the methodology to the effective measurement of diversification in multi-asset portfolios invested in traditional assets (stocks and bonds) as well as real assets (commodities and real estate), and we stress that an equally weighted portfolio, while being perfectly diversified in terms of asset class dollar contributions, is not necessarily well diversified when it comes to factor risk contributions. We also explore the properties of *factor risk parity* portfolios, which are formally defined as portfolios in which factors equally contribute to risk [Deguest et al. 2013]. Because there is no unique portfolio satisfying the factor parity condition, we also introduce various criteria for narrowing down to one the number of eligible portfolios. Finally, we discuss the use of factor risk budgeting constraints in multi-asset portfolio construction and provide empirical evidence that imposing a minimum ENB is an effective way to improve diversification with respect to standard mean–variance analysis.

FACTORS IN MULTI-ASSET PORTFOLIOS

The main focus of this section is to provide an actionable taxonomy of factors, as well as some conceptual clarification with respect to how the various types of factors may be used in the context of multi-asset solutions. In investment practice, the notion of factor is actually used with many different meanings—a situation that often leads to confusion, and sometimes disappointment, about the potential benefits of factor investing. Before analyzing the role that factors can play in the design of multi-asset investment solutions, we first discuss three distinct conceptions of factors that can be of practical relevance—namely, factors defined as profitable systematic strategies, factors defined as business-cycle-related variables, or factors defined as common sources of risk. Note that these categories do not form a mutually exhaustive partition of the set of factors. For example, factors such as the value and size factors in equity markets can be regarded as both rule-based strategies that deliver a positive expected excess return in the long run and common sources of risk that can be used to decompose an equity portfolio volatility or tracking error.

Factors as Profitable Strategies, State Variables, or Common Sources of Risk

First, the term *factor*, or *rewarded factor*, is often used to refer to a strategy that earns a premium in the long run without relying on active management techniques. Not only must the premium show up in historical data, but it must also be backed by some intuitive justification related to undiversifiable risk, market frictions, or behavioral biases, to justify its robustness and persistence in the future. This definition of factors is particularly relevant in the equity class, where vast empirical and theoretical literature can provide guidance on the robust identification of rewarded sources of risk. The size, value, momentum, low-volatility and quality effects are among the most consensual of these empirical regularities,[1] and a number of so-called *smart* factor indices have been launched by index providers as benchmark portfolios for the extraction of the associated risk premia.

On the other hand, the list of historically proven and economically justified sources of long-term profitability in a multiclass universe is comparatively shorter. Recent research has identified some patterns that seem to span several asset classes—such as long-term reversal and short-term momentum in sovereign bonds, commodities, and currencies (Asness et al. [2013] and Jostova et al. [2013]), carry effects (Koijen et al. [2013]), as well as low-beta effects in equities, Treasuries, credits, commodities, and currency futures (Frazzini and Pedersen [2014]). However, this list is likely incomplete, and many of these factors have not yet been thoroughly investigated or adopted in investment practice. In the absence of a consensual inventory of sources of premia and of cost-efficient replicating investment vehicles, the definition of a factor as a profitable strategy does not seem well suited to the analysis of multi-asset allocation problems.

A more relevant notion of *factor* in a multi-asset context is that of a *state variable* that contributes to define market conditions by being a determinant of the time-varying expected returns and volatilities of asset classes (or rewarded factors within asset classes). Well-known examples are the dividend yield, which is a traditional predictor of stock returns (Fama and French [1989]), or forward-spot spreads, which have predictive power for bond returns (Fama and Bliss [1987] and Cochrane and Piazzesi [2005]). In GARCH models (Bollerslev [1986]), the current volatility level and the current innovation in returns are also factors because they are used as predictors of future volatility. Merton's [1973] intertemporal capital asset pricing model makes a connection between the time series and the cross-section perspectives by showing that exposures to the state variables that capture time variation in expected returns, volatilities, and correlations account for the differences in expected returns across assets. Macro factors are also natural candidates for defining meaningful regimes because an asset's price can be thought of as the expected value of discounted future cash flows, so that asset returns are impacted by inflation, output, and interest-rate-related variables through the discount rate or expected future income.

As a simple approach to gauging the potential impact of these variables on asset allocation decisions, we define regimes based on their values and evaluate the conditional average returns, volatilities, and correlations of asset classes. If we find significant variation in parameters across regimes, this strongly suggests that investors should take into account the current state of the economy when designing a multi-asset allocation.[2] Beyond the practical difficulties involved in implementation, it should however be recognized that this otherwise useful notion of factors does not necessarily allow for a better measurement or management of diversification in multi-asset portfolios within each regime.

The last definition, in which a *factor* is regarded as a *risk factor* (that may or may not be rewarded)—that is, a common source of risk—is perhaps the most relevant for multi-asset portfolio management. Such risk factors are said to be *explicit* if their values are directly observed, and *implicit* when the factors have to be extracted from asset returns. Explicit factors have the obvious advantage that they are interpretable, especially when they are business cycle variables, because such variables can be related to general sources of uncertainty. Unfortunately, macro factors often explain a small fraction of the variance of returns in equity portfolios and in multi-asset portfolios (see, e.g., Table III of Asness et al. [2013]). Implicit factors have by construction much larger explanatory power because they are designed to capture all sources of risk in a set of asset returns.

For the purpose of identifying nonoverlapping underlying sources of risk, which will be argued below to be a key ingredient in the measurement of diversification for multi-asset portfolios, it is often convenient to use uncorrelated factors. Principal component analysis (or PCA) is arguably the most commonly used approach, but it suffers from a number of shortcomings. Carli et al. [2014] point that principal factors are often hard to interpret and are unstable across different time periods; and Meucci et al. [2015] show that if all assets have the same volatility and pairwise correlation, then an equally weighted portfolio is fully invested in the first principal factor (i.e., the one with the largest variance), while the other factors have zero weight.

As a result, portfolio risk is entirely explained by the first factor, regardless of the value of the common correlation. This property is counterintuitive, as one would expect uncorrelated factors to have more balanced contributions when the interasset correlation shrinks to zero. To address these issues, Meucci et al. [2015] introduce a competing method known as *minimum linear torsion* (or MLT), the objective of which is to extract N uncorrelated linear combinations of assets that minimize the distance with respect to the original assets, as measured by the sum of the squared tracking errors between the factors and the assets. In other words, minimum linear torsion is the method of extracting uncorrelated factors that least distorts the original assets. In what follows, we present an empirical illustration of the PCA and MLT techniques for extracting implicit risk factors in a multi-asset universe.

Extracting Implicit Risk Factors in a Multi-Asset Universe

In this empirical illustration, we consider seven asset classes, including U.S. equities represented by the S&P 500 Index, world non-U.S. equities represented by the MSCI World Ex-U.S. Index, U.S. Treasuries (the Barclays U.S. Treasury Index), U.S. corporate bonds (the Barclays U.S. Baa Index), U.S. Treasury Inflation-Protected Securities (the Barclays U.S. TIPS Index), commodities (the S&P GSCI Index) and real estate (the Dow Jones U.S. Select REIT Index). The data is monthly and covers the period from April 1997 to September 2017. Index series are converted to excess returns by subtracting the U.S. Treasury bill secondary market rate of three-month maturity. Principal component analysis and minimum linear torsion techniques are successively applied to extract seven implicit uncorrelated risk factors.

Exhibit 1 shows the composition of the statistical factors in terms of assets. Because the variables analyzed are excess returns, the coefficients shown in the figure can be interpreted as the weights of long–short portfolios invested in the constituents. In the MLT procedure, each factor is intended to be a proxy of the corresponding asset—in fact, the closest proxy subject to the constraint that the factors must be pairwise uncorrelated. Panel B of Exhibit 1 clearly shows that each factor is mainly composed of the asset to which it is associated. The positions in the other assets often have negative signs in order to eliminate the effects of the interasset correlations. For example, considering Factors 1 and 2, which respectively correspond to U.S. equities and world equities, the short position in the other equity index is sizable for each one of the two equity indices, because they have a correlation of 86.8% in the sample period.

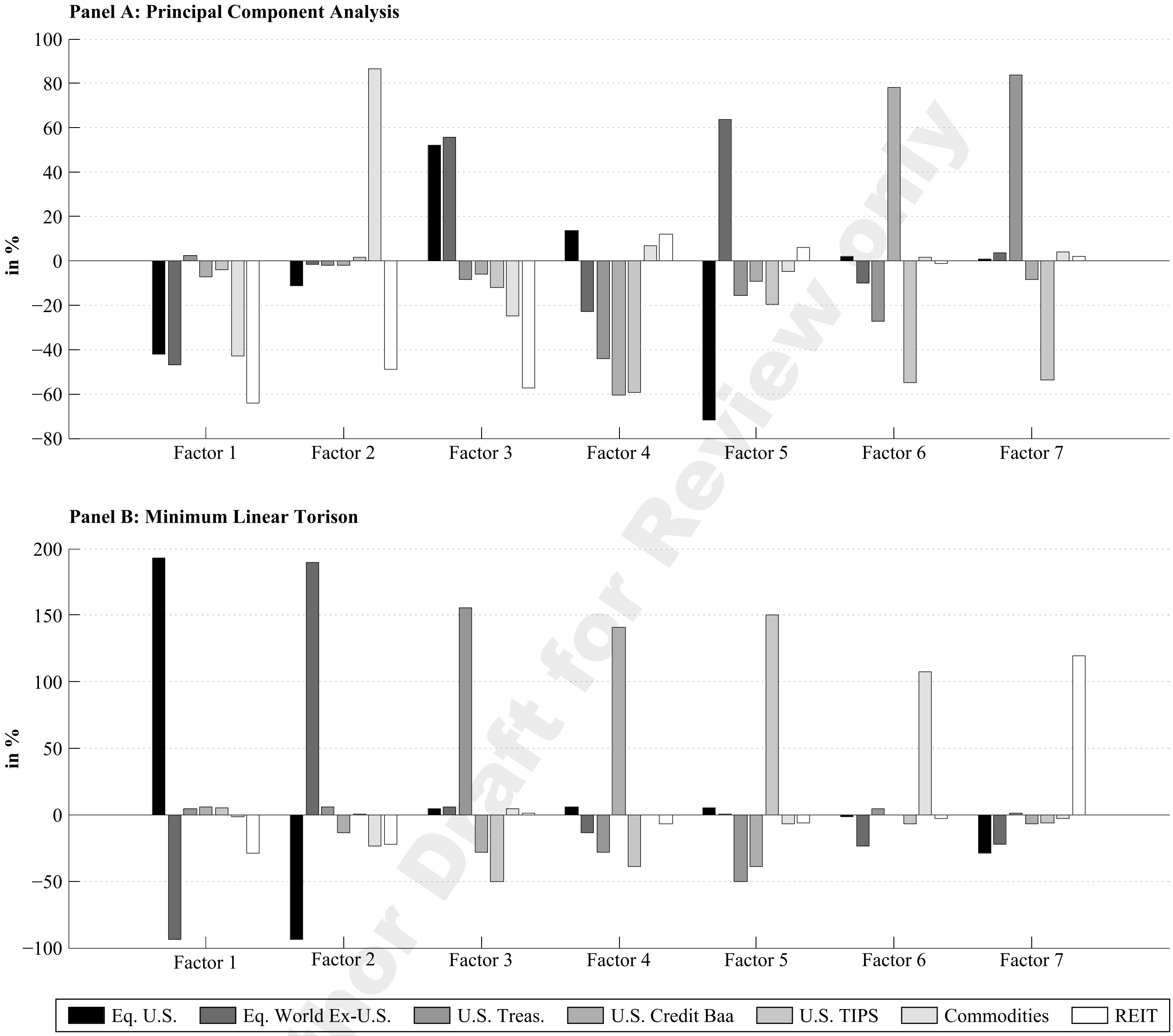
The interpretation of principal components (PCs) is not as straightforward as for MLT factors. Panel A of Exhibit 1, which displays the composition of each factor, shows that Factor 1 loads negatively on almost all asset classes (except for U.S. Treasuries) but has very small loadings on the three bonds. It can be regarded as representing some kind of proxy for an economic growth market factor. Factor 2 loads almost exclusively on commodities and real estate with opposite signs, so it could be described as a factor capturing the net return difference between commodities and real estate.[3] Factor 3 is related to the spread between equities and real assets classes, because it loads positively on both equity indices and takes short positions in commodities and real estate.

In contrast with the previous factors, which have much lower exposures to bonds than to other classes, Factor 4 is mostly exposed to the three bond indices. It therefore accounts for the commonalities between the three bond indices and thus corresponds to an interest-rate-level factor. Factor 5 is mainly exposed to equities and captures the differences between the United States and the rest of the world; as such, it can be regarded as an international equity factor. The last two factors, Factor 6 and Factor 7, correspond to the return spread between corporate bonds and TIPS for Factor 6, and Treasuries and TIPS for Factor 7. Factor 7 is possibly related to the spread between nominal and real interest rates, which is also known as the breakeven inflation rate and includes contributions from expected inflation and the inflation risk premium.

Exhibit 2 provides another perspective by showing which factors are important to explaining time variation in the returns of a given asset. We conduct this exercise only for PCA factors because MLT factors are constructed to provide a direct answer to this question. For each asset, we compute the tracking error with respect to a linear combination of one, two, three, four, five, or six factor(s), the coefficients being the exposures of

E X H I B I T 1

Composition of Implicit Risk Factors in Terms of Assets, 1997–2017



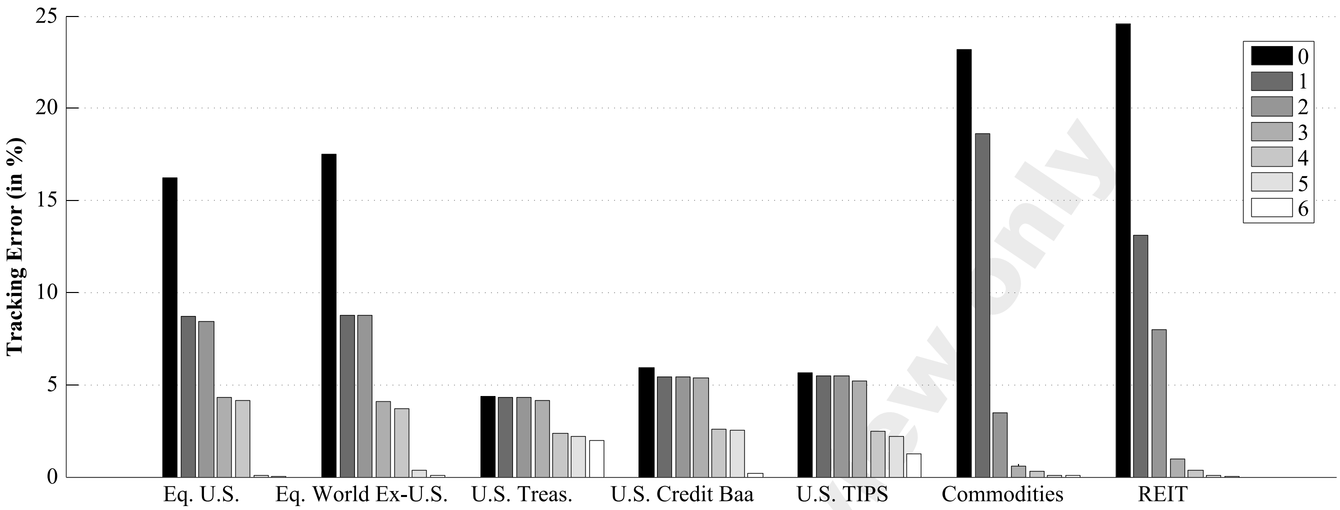
Notes: Each picture shows the composition of implicit uncorrelated risk factors in terms of the seven asset classes analyzed: U.S. equities, world ex-U.S. equities, U.S. Treasuries, U.S. corporate bonds, U.S. TIPS, commodities, and real estate. In Panel A, factors are extracted by principal component analysis, whereas in Panel B, they are obtained through minimum linear torsion. The statistical procedures are applied on monthly excess returns.

the asset with respect to the factors. We also report the tracking error with respect to a portfolio with no factor (i.e., a constant) that is the volatility of the asset class. These combinations can be regarded as factor portfolios proxying for an asset class with increasing accuracy.

The tracking error is, by construction, exactly zero when all seven factors are included in the replicating portfolio. We see that eliminating the influence of the first factor significantly reduces volatility for all classes except for the three bond indices. This is consistent

EXHIBIT 2

Approximation of Asset Excess Returns with Linear Combinations of Principal Factors, 1997–2017



Notes: Principal factors are extracted by analyzing the monthly excess returns to seven asset classes over the period from April 1997 to September 2017: U.S. equities, world ex-U.S. equities, U.S. Treasuries, U.S. corporate bonds, U.S. TIPS, commodities, and real estate. Asset excess returns are then proxied with linear combinations of one to seven factor(s), and the figure shows the annualized tracking error of the factor portfolio with respect to the asset. Bars 0 show asset volatilities over the sample period, and Bars 7 have zero height because each asset is perfectly replicated by a portfolio of the seven factors.

with the interpretation of Factor 1 as a broad economic growth factor. The next factor, Factor 2, is only useful in explaining commodities and real estate. For all other classes, the replication quality is hardly improved by moving from one to two factor(s). Factor 3 brings an improvement for all classes except bonds, and only the factors with significant exposures to bonds in Exhibit 1, namely Factor 4 and Factor 6, generate a decrease in the tracking error for bonds. With six replication factors, all tracking errors are virtually zero, and the marginal contribution of Factor 7 is hardly noticeable.

In closing, note that we can explain the fact that interest rate risks do not materially impact the first factors by the low volatilities of the bond indices with respect to the other classes: Over the sample period, the volatility of excess returns is 16.2% for U.S. equities, 17.5% for international equities, 23.2% for commodities, and 24.6% for real estate, but only 4.4%, 5.9%, and 5.7% respectively for Treasury bonds, corporate bonds, and TIPS. Being far less volatile than the other asset class indices, bond indices represent only a small fraction of the aggregate risk; and the first three factors from the PCA focus on the statistical explanation of the other classes, explaining 93.9% of the aggregate risk in this example. Although this is a large fraction, and although these three factors are adequate at describing equities, commodities, and real estate, they are close to useless when it comes to explaining bonds returns, and Factors 4, 6, and 7 are critically needed to generate a good bond-replicating portfolio. This example shows that not all economically important sources of risk—here, interest rate risk—are captured by the first factors in a principal component analysis.

MEASURING DIVERSIFICATION FOR MULTI-ASSET PORTFOLIOS

Before we move to the design of a framework suitable for the construction of well-diversified portfolios, we must be able to measure the level of diversification of a given portfolio. In this section, we argue that implicit risk factors are particularly well suited for this purpose, given that they comprehensively capture the sources of risk behind a given set of assets.

Effective Number of Constituents versus Effective Number of Bets

The nominal number of constituents in a portfolio is a grossly misleading measure of diversification because it does not recognize that some constituents can have much larger weights. For example, a hundred-asset portfolio invested 99% in the first asset can hardly be regarded as well-diversified, even if the nominal number of assets is large. A finer measure is the effective number of constituents (ENC), which quantifies the level of diversification by measuring how dollars are spread across constituents. It is defined as the reciprocal of the Herfindahl index, which is the sum of squared weights. For a portfolio with N constituents and weights $w_1, \ldots, w_N$, we have

$$ENC = \frac{1}{\sum_{i=1}^{N} w_i^2} \tag{1}$$

By using the Cauchy–Schwartz inequality, we can show that the ENC is always less than or equal to N, a value attained only by the equally weighted portfolio. So, under the ENC measure, the equally weighted portfolio is the best diversified portfolio, albeit in a naive sense. Equal weighting is equivalent to Sharpe ratio maximization when all assets have the same expected returns, volatilities, and pairwise correlations. In other words, the $1/N$ portfolio is justified as a proxy for the maximum Sharpe ratio portfolio under the assumption that all constituents are indistinguishable.

In general, the ENC measure, while better than the nominal number of constituents measure, can still be misleading because it completely ignores the risk characteristics of constituents, in particular the fact that some may be far more volatile than others. It is well known (Qian [2005]) that in an equally weighted portfolio of stocks and bonds, a large fraction of the portfolio volatility is due to the equity component of the portfolio. For the purpose of the illustration, assume that stock and bond volatilities are respectively 20% and 5% per year and that they have zero correlation. Then, the ex ante volatility is

$$\sigma = \sqrt{0.5^2 \times 0.2^2 + 0.5^2 \times 0.05^2} = 10.3\% \tag{2}$$

and the percentage contribution of the equity component is

$$\frac{0.5^2 \times 0.2^2}{\sigma^2} = 94.1\% \tag{3}$$

while that of the fixed-income component is 5.9%. To fix this problem, we introduce a new measure, dubbed the *effective number of correlated bets* (ENCB), which is defined as an ENC measure applying to risk as opposed to dollar percentage contributions. For the equally weighted stock-bond portfolio in our stylized example, we have

$$ENCB = \frac{1}{0.941^2 + 0.059^2} = 1.1 \tag{4}$$

which is much smaller than the ENC of 2. Note that the ENCB measure is maximized by the risk parity portfolio, in which all constituents have the same contribution to volatility. In the two-asset case, the weights are proportional to the reciprocals of volatilities. With more than two constituents, no algebraic formula is known for the risk parity weights, and numerical methods must be employed (Maillard et al. [2010]).

One key shortcoming of the ENCB measure is that it does not recognize that a portfolio of assets with similar correlations and volatilities can hide an extremely concentrated set of factor exposures. Consider for instance an equally weighted portfolio of two bonds with similar durations, hence similar volatilities. Both bonds have about the same contribution to overall volatility, so that the ENCB is close to 2, but the portfolio is almost exclusively exposed to one risk factor, which is the level of interest rates.

To overcome this limitation of the ENCB, Meucci [2009a] and Deguest et al. [2013] suggest evaluating a portfolio's diversification level by measuring the contributions of risk factors extracted by principal component analysis or minimum linear torsion. Formally, consider a set of N uncorrelated risk factors, given as nonredundant linear combinations of asset returns. Then, asset returns can be recovered from factor values by inverting the system of equations that expresses the factors in terms of assets. The return to a portfolio, which is the weighted sum of asset returns, can be rewritten as a sum of factor returns and, given the absence of correlation across factors, the portfolio variance can be written as

$$\sigma_p^2 = \sum_{k=1}^{N} \beta_{kp}^2 \, \sigma_{F,k}^2 \tag{5}$$

where the β_{kp} are the portfolio's factor exposures and the $\sigma_{F,k}^2$ are the factor variances. From this equation, it is straightforward to define the percentage contribution of each factor as

$$c_{F,k} = \frac{\beta_{kp}^2 \sigma_{F,k}^2}{\sigma_p^2} \tag{6}$$

The degree of diversification in terms of underlying risk factors is measured by the *effective number of uncorrelated bets*:

$$ENB = \frac{1}{\sum_{k=1}^{N} c_{F,k}^2} \tag{7}$$

Going back to the eggs-and-baskets analogy, the ENB measures diversification in a setting in which eggs are risk contributions and baskets are uncorrelated risk factors as opposed to being correlated asset classes: a large ENB reflects the proverbial definition for diversification in that it means that eggs are well spread across baskets.

Empirical Illustration: Assessing Diversification in Terms of Factors

The effective number of uncorrelated bets (ENB) depends on the chosen risk factors. In Exhibit 3, we compute the ENB measure for five multi-asset portfolios based on the two competing methods, namely principal component analysis and minimum linear torsion. The first portfolio is a policy portfolio with a 60% weight in equities, 30% in bonds, and 10% in alternative assets. Within each group, the portfolio is equally split across constituents, so there is a 30% weight in U.S. equities, 30% in international equities, 15% in U.S. Treasuries, 15% in U.S. credits, 3.33% in TIPS, 3.33% in commodities, and 3.33% in real estate. The second portfolio is equally weighted across the seven classes. The third and the fourth ones are the long–short and the long-only versions of the global minimum-variance portfolio, and the last one is a standard risk parity portfolio (Maillard et al. [2010]). The weights in percentage points of the last three portfolios, which are computed over the full sample, are given by

$$\begin{aligned} w_{GMV-LS} &= [12.0 \;\; -0.1 \;\; 90.5 \;\; 4.7 \;\; -8.0 \;\; 3.2 \;\; -2.4] \\ w_{GMV-LO} &= [10.0 \;\; 0 \;\; 87.1 \;\; 0 \;\; 2.9 \;\; 0 \;\; 0] \\ w_{RKP} &= [7.4 \;\; 6.3 \;\; 40.6 \;\; 16.4 \;\; 18.5 \;\; 6.2 \;\; 4.6] \end{aligned} \tag{8}$$

All three portfolios overweight Treasury bonds, which form the least-volatile class, but the risk parity one is less concentrated than the two minimum-variance allocations, because the constraint of equal risk contributions precludes the corner solutions that are common in optimized portfolios. Exhibit 3 also reports several standard risk and performance measures.

Exhibit 3

Effective Number of Uncorrelated Bets with Implicit Risk Factors and Other In-Sample Statistics for Benchmark Portfolios, 1997–2017

	Policy	Equal Weight	Min. Vol. (long–short)	Min. Vol. (long-only)	Risk Parity
Average Ret. (%)	7.2	6.6	5.2	5.3	5.9
Volatility (%)	10.9	9.7	3.8	3.9	5.2
Sharpe Ratio	0.47	0.47	0.83	0.85	0.74
Max. DD (%)	40.4	38.5	5.3	5.9	16.9
ENC	4.38	7.00	1.19	1.30	4.14
ENCB	2.52	4.53	1.19	1.30	7.00
ENB (PCA)	1.34	1.08	2.56	2.67	2.00
ENB (MLT)	3.40	3.77	2.11	2.28	6.00

Notes: Implicit uncorrelated factors are extracted from the monthly excess returns to seven asset classes over the period from April 1997 to September 2017: U.S. equities, world ex-U.S. equities, U.S. Treasuries, U.S. corporate bonds, U.S. TIPS, commodities, and real estate. The table displays the in-sample average return, volatility, Sharpe ratio, and maximum drawdown, as well as the effective number of constituents, correlated bets, and uncorrelated bets for five portfolios: a policy portfolio with 60% in equities, 30% in bonds, and 10% in alternative assets; the equal-weighted portfolio; the long–short and the long-only minimum-variance portfolios; and the risk parity portfolio. Principal component analysis and minimum linear torsion are applied to extract implicit factors.

With seven factors, the maximum possible ENB is 7. The portfolio that is closest to this limit is the risk parity one when MLT factors are employed, with an ENB of 6. Indeed, each MLT factor is "close" to a constituent, so that a portfolio that equates risk contributions across assets also has a small dispersion of contributions across factors: A risk parity portfolio can be regarded as a proxy for a "factor risk parity" portfolio—an allocation scheme that we discuss in detail in the next section of this article.

The objectives of spreading contributions across assets or across factors conflict much more with each other when diversification is measured with respect to PCA factors. In particular, long-only portfolios are inevitably dominated by the first factor, so that factor diversification is harder to achieve. For example, the equally weighted portfolio has an ENB of 1.08 only. The largest ENBs, at 2.56 and 2.67, are attained by the two minimum-volatility portfolios. The higher level of diversification of these portfolios across factors stands in contrast with their high concentration in Treasuries, which should intuitively penalize their diversification level. This happens because the large weight in Treasuries re-equilibrates the allocation toward interest rate risk factors, which are marginal from a statistical standpoint because they correspond to principal factors 4, 6, and 7. Equity factors are overweighted with respect to the other factors in the policy and the equally weighted portfolios, which reduces their ENB. As a general rule, the ENB is very sensitive to the choice of the underlying factors, so the set of factors should always be specified together with an ENB value. Another general remark is that using PCA implies a low ENB almost by construction because the largest possible fraction of asset return variance will be explained by a limited number of factors, and we therefore recommend using MLT when measuring diversification for multi-asset portfolios.

MANAGING DIVERSIFICATION FOR MULTI-ASSET PORTFOLIOS

This last section explores the construction of multi-asset portfolios that are better diversified in terms of factor contributions than those listed in Exhibit 3 through the introduction of ENB in the portfolio construction methodology, either as an optimization objective—in order to achieve the highest level of diversification (as in spreading eggs, or risk contributions, in as many untied baskets as possible)—or as a constraint—in order to optimize another criterion, for example, to minimize risk without sacrificing diversification across factors.

From Risk Parity to Factor Risk Parity Portfolios

Factor contributions sum up to 1, so the Cauchy–Schwartz inequality implies that the ENB of an N-asset portfolio is always less than or equal to N. The maximum is attained when all factors contribute equally to risk, a property that defines a *factor risk parity* (FRP) portfolio. Given the expression for factor contributions in Equation 6, this condition says that the squared portfolio exposures must be proportional to the inverse of factor variances. Because factors are linear combinations of asset returns, $F = A'R$, it is possible to revert back from each set of factor exposures to asset weights by writing

$$w_i = \sum_{k=1}^{N} A_{ik} \beta_{kp} \tag{9}$$

Taking the betas to be inversely proportional to volatilities, this equation gives the composition of an FRP portfolio.

The factor risk parity condition does not define a unique portfolio. Deguest et al. [2013] actually show that for N assets, there are up to 2^{N-1} distinct FRP portfolios. The intuitive argument is as follows. Factor contributions are equal if, and only if, the portfolio's squared factor exposures are inversely proportional to factor variances—that is, if the absolute values of exposures are inversely proportional to factor volatilities. But while it is natural to exclude short positions in assets, there is ex ante no reason why a portfolio should have long-only exposures to the risk factors, so that the sign of each exposure is a free parameter. Because exposures are in fact restricted by the budget constraint (asset weights must add up to 1), only $N - 1$ signs can be fixed independently from each other. Because there are two possible signs for each exposure, this leads to 2^{N-1} possible combinations of signs and therefore, at most, 2^{N-1} different FRP portfolios. Appendix A gives the mathematical details of the argument.

These FRP portfolios can be distinguished through various criteria, and one can choose one of them by optimizing a criterion subject to the parity constraint.

For example, one could focus on the "FRP minimum variance" and the "FRP maximum Sharpe ratio" respectively, defined as the FRP portfolios that minimize volatility and maximize Sharpe ratio. Another sensible selection criterion in practice is based on the examination of the degree of leverage. Because many of the FRP portfolios involve short positions in assets classes, investors in multi-asset investment solutions may choose the one that minimizes the magnitude of these positions. Imposing leverage constraints also has the advantage of mitigating the effects of estimation errors: This is an interesting property because implicit factors are estimated rather than observed.

In general, FRP portfolios are not efficient portfolios in the usual mean–variance sense. On the other hand, it can be shown that the maximum Sharpe ratio portfolio coincides with one of the FRP portfolios if, and only if, the Sharpe ratios of the factors (regarded as long–short portfolios of assets) have the same absolute value (see the Appendix B for a proof).

As opposed to maximizing the ENB by constructing an FRP portfolio, it is also possible to use this measure to introduce a diversification constraint across factors in an optimization program. Such a constraint is similar in spirit to the "norm constraints" on weights introduced by DeMiguel et al. [2009]: Indeed, a norm constraint can be rephrased as a minimum ENC constraint, which aims to ensure that the optimized portfolio is sufficiently diversified across its constituents. As DeMiguel et al. [2009] show, such a constraint leads to lower out-of-sample volatility and higher out-of-sample Sharpe ratio for estimated minimum-variance portfolios, even though the true unknown minimum-variance portfolio does not satisfy it. This is yet another example of the usefulness of "wrong" weight constraints in the presence of parameter uncertainty, as in Jagannathan and Ma [2003].

Following this principle, a minimum ENB constraint can be introduced in various optimization programs to ensure that risk contributions are not overly concentrated in a few factors and to reduce inherent biases in some weighting schemes. For instance, an unconstrained minimum-variance portfolio of stocks, Treasury bonds, and other asset classes is always highly biased toward Treasuries, which are the least-volatile asset class, and is thus dominated by interest rate risk. A minimum ENB constraint would allow for a minimum level of factor exposure diversification within multi-asset portfolios.

Constructing Factor Risk Parity Portfolios

For seven constituents, the number of portfolios that satisfy the factor risk parity condition is $2^{7-1} = 64$, so it is feasible to compute all of them, using the formulas given in Deguest et al. [2013] and recalled in Appendix A. All of them involve short positions in some constituents, sometimes sizable, The total leverage, defined as the negative of the sum of negative weights, is plotted in Panel A of Exhibit 4 and varies considerably from one portfolio to the other. Some portfolios have exceedingly high levels of leverage—as high as 27, or 2,700%—whereas the minima are much more reasonable—at 45.0% for PCA factors and 0.4% for MLT factors. The corresponding weights (expressed in percentage points) are

$$\begin{aligned} w_{PCA,LEV} &= [13.7\ \ -11.7\ \ 98.9\ \ -33.3\ \ 12.0\ \ 15.7\ \ 4.7] \\ w_{MLT,LEV} &= [15.0\ \ 4.3\ \ 42.1\ \ 18.2\ \ 15.5\ \ 5.3\ \ -0.4] \end{aligned} \tag{10}$$

With MLT factors, the FRP portfolio that minimizes leverage is almost a long-only one. It can be turned into a strictly long-only portfolio by setting the real estate weight equal to zero and rescaling the other weights to bring their sum back to 100%. The composition of this "rescaled" portfolio is

$$w_2 = [14.9\ \ 4.3\ \ 41.9\ \ 18.1\ \ 15.4\ \ 5.3\ \ 0] \tag{11}$$

and its ENB with MLT factors is 6.99, which is indeed very close to 7. In details, the percentage contributions to risk of the MLT factors are

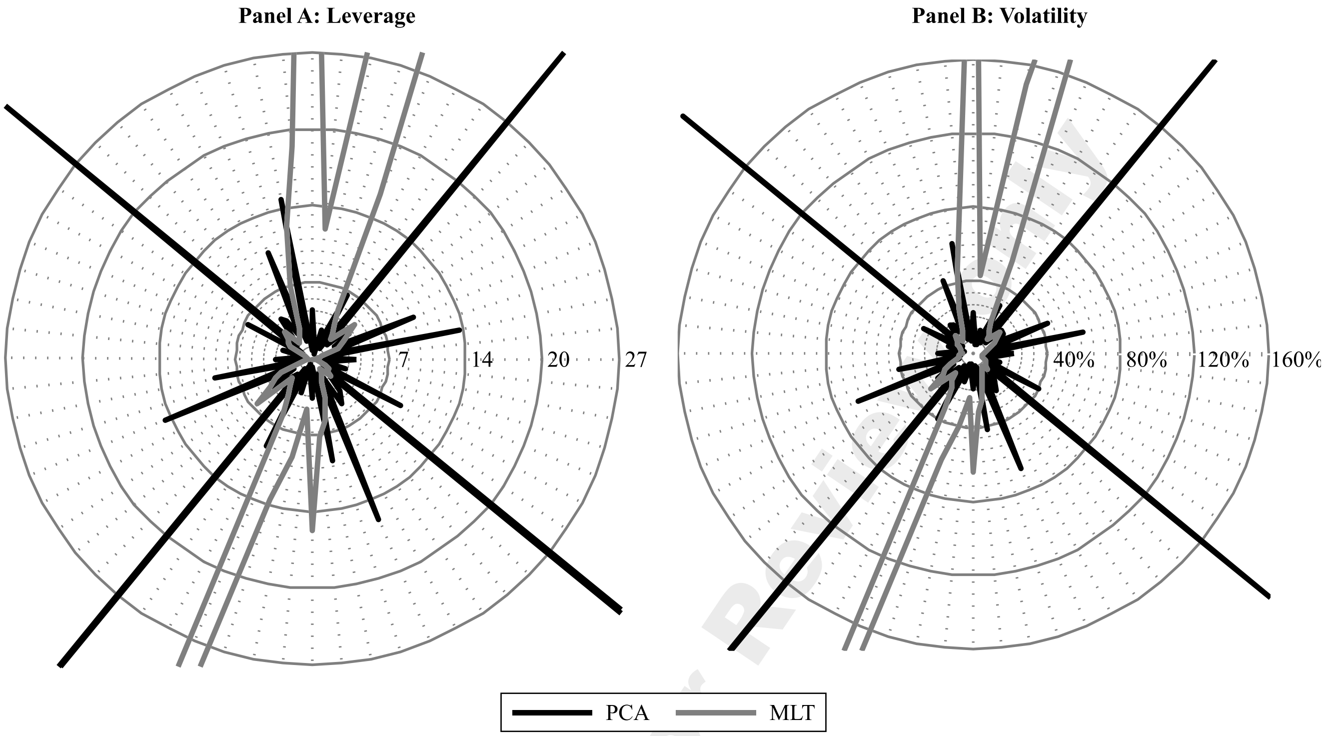
$$[14.3\ \ 14.3\ \ 13.9\ \ 14.0\ \ 14.0\ \ 14.1\ \ 15.4] \tag{12}$$

The dispersion is small, so the long-only portfolio w_2 is a good approximation for a factor risk parity portfolio.

Panel B displays the ex ante volatilities of the 64 portfolios: the ex ante volatility is the quantity $\sqrt{w'\Sigma w}$, where w denotes the vector of weights and Σ is the covariance matrix. Rather unsurprisingly, volatility turns out to be strongly related to leverage, as indicated by the similar shapes of the two graphs. In fact, whether PCA or MLT factors are employed, the FRP portfolio that minimizes the variance is also the one that minimizes the sizes of the short positions. It has an ex ante

E X H I B I T 4

Total Leverage and Ex Ante Volatility of Factor Risk Parity Portfolios, 1997–2017



Notes: Implicit factors are extracted by analyzing the monthly excess returns to seven asset classes over the period from April 1997 to September 2017: U.S. equities, world ex-U.S. equities, U.S. Treasuries, U.S. TIPS, U.S. corporate bonds, commodities, and real estate. Principal component analysis and minimum linear torsion are applied to extract uncorrelated risk factors, and the 64-factor risk parity portfolios equate the contributions of the implicit factors to portfolio volatility. Panel A shows the total leverage of each portfolio, defined as the sum of the absolute values of the negative positions, and Panel B shows the annualized ex ante volatilities of the 64 portfolios. For readability, portfolios with a leverage greater than 27 (8 out of 64) or a volatility greater than 160% (7 out of 64) are not represented.

volatility of 5.0% per year with PCA factors and 4.9% with MLT factors. For comparison purposes, the ex ante volatility of the minimum-variance portfolio subject to a long-only constraint is 3.8%. Again, we would recommend the use of MLT factors both for enhanced robustness and ease of interpretation.

Introducing Factor Risk Budgeting Constraints

The factor risk parity allocation rule ensures by definition an equal contribution of all factors to portfolio risk, but it does not include a control for other dimensions, such as the ex ante volatility. Moreover, considering that underlying risk factors are not observed and are only estimated through statistical procedures, it may be wise not to rely entirely on these statistical constructs and to preserve a sufficient level of diversification across assets. This section focuses precisely on the design of allocations in which diversification across factors is not the main objective to pursue but is introduced as a control to deconcentrate portfolios that have otherwise no reason to spread contributions across factors.

The starting point is the set of optimized portfolios from Exhibit 3, all of which optimize a criterion but most of which have an ENB lower than 3.5—that is, less than half the maximum possible value. We then perform ENC maximization, volatility minimization, and ENCB maximization subject to the constraint that the ENB is at least equal to 75% of the maximum possible value. When short sales are permitted, the maximum ENB is 7—that of a factor risk parity portfolio—and the minimum ENB for the optimized portfolios is set to be $0.75 \times 7 = 5.25$. If short sales are excluded, the maximum

EXHIBIT 5
Properties of Portfolios Subject to Minimum ENB Constraint, 1997–2017

	Long–Short				Long-Only			
	Max. ENB	**Min. Vol.**	**Max. ENC**	**Max. ENCB**	**Max. ENB**	**Min. Vol.**	**Max. ENC**	**Max. ENCB**
PCA Factors								
Average (%)	45.0	24.8	19.2	440.0	0.0	0.0	0.0	0.0
Avg. Ret. (%)	4.6	4.9	5.0	6.0	4.6	5.1	4.8	5.2
Volatility (%)	5.1	4.2	5.5	20.7	4.9	4.1	5.9	5.0
Sharpe Ratio	0.51	0.69	0.54	0.19	0.52	0.75	0.47	0.62
Max. DD (%)	11.1	8.0	15.3	74.4	13.2	8.1	19.9	14.1
ENC	0.86	0.99	2.37	0.05	1.48	1.46	3.83	2.62
ENCB	1.99	1.51	2.73	6.91	2.22	2.09	2.70	3.89
ENB	**7.00**	5.25	5.25	5.25	**4.87**	3.66	3.66	3.66
MLT Factors								
Leverage (%)	0.4	2.1	0.0	0.0	0.0	0.0	0.0	0.0
Average Ret. (%)	5.8	5.6	6.1	5.9	5.8	5.6	6.1	5.9
Volatility (%)	4.9	4.1	6.9	5.2	4.9	4.2	6.9	5.2
Sharpe Ratio	0.77	0.85	0.59	0.74	0.77	0.86	0.59	0.74
Max. DD (%)	15.3	9.1	26.7	16.9	15.4	9.4	26.7	16.9
ENC	3.82	2.48	5.90	4.14	3.81	2.41	5.90	4.14
ENCB	4.86	3.41	5.27	7.00	4.98	3.56	5.27	7.00
ENB	**7.00**	5.25	5.25	6.00	**6.99**	5.25	5.25	6.00

Notes: Implicit uncorrelated factors are extracted by principal component analysis (PCA) or minimum linear torsion (MLT) from the returns to seven asset classes: U.S. equities, world ex-U.S. equities, U.S. Treasuries, U.S. corporate bonds, U.S. TIPS, commodities, and real estate. We compute a maximum ENB portfolio (approximation for a factor risk parity portfolio), a maximum ENC (approximation for an equally weighted portfolio), a minimum volatility, and a maximum ENCB (approximation for a risk parity portfolio) subject to a minimum ENB constraint; the minimum ENB is 75% of the maximum possible ENB (in bold face). The exercise is done with and without allowance for short sales.

feasible ENB is lower than 7, and we estimate it by maximizing the ENB subject to long-only constraints; the result is 6.99 for MLT factors—a value almost equal to 7 because there is an FRP portfolio that is almost long-only. For PCA factors, the maximum, 4.87, is of course much lower than 7, because long-only portfolios are dominated by the first factor. The minimum ENB required for the other portfolios is therefore set to the value $0.75 \times 4.87 = 3.66$.

Exhibit 5 presents standard risk and performance indicators for the portfolios optimized over the full sample, together with the three diversification measures ENC, ENCB, and ENB. The ENB constraint leads to allocations that are more diversified across factors than the unconstrained portfolios in Exhibit 3. For the maximum ENCB portfolio, the constraint is not binding, because the unconstrained version already has an ENB of 6, thus greater than 5.25. For the minimum-volatility portfolio, it is interesting to note that the constraint improves the ENB with respect to the unconstrained case (in which the ENB is 2.67 or 2.28, depending on the set of factors used) without excessively damaging volatility, which is 3.9% for the unconstrained long-only allocation, and 4.0% or 4.2% with the constraint.

For PCA factors, the better diversification is achieved at the expense of substantial concentration in some assets, as illustrated by the high-leverage levels of long–short portfolios and the low ENCs of long-only portfolios—the best score being only 3.83. The short positions can be massive: In the case of the maximum ENCB portfolio, they represent 440% of the portfolio value, leading to a strong negative impact on the portfolio Sharpe ratio (0.19 only) and maximum drawdown (74.4%). Long-only constraints mitigate this problem, but diversification across factors must still be traded against diversification across assets, and the final compromise is a relatively low score on both sides because the maximum ENCB portfolio has an ENCB of only 3.89 and an ENB of only 3.66.

Without the ENB constraint, however, the compromise was heavily biased toward assets, with an ENCB of 7 and an ENB of 2. For the maximum ENC portfolio, the ENB constraint has an interesting potential to reduce the maximum drawdown, from 38.5% in the unconstrained case (see Exhibit 3) to 19.9% here. A reduction is also observed for the maximum ENCB allocation, but it is less spectacular, from 16.9% to 14.1%. These results suggest that benefits can be expected from the simultaneous diversification of risk across factors and across assets. This makes intuitive sense, given that in the presence of uncertain parameters (here, uncertain implicit factor values), preserving some level of diversification across assets improves the robustness of the portfolio construction method (here, diversification across factors).

With MLT factors, the two diversification objectives (across assets and across factors) are easier to reconcile: long–short portfolios have less leverage and the long-only ones have higher ENCs than the allocations based on PCA factors. Some portfolios achieve attractive diversification levels from the perspective of the three indicators. For example, the maximum ENC portfolio has an ENC of 5.90, an ENCB of 5.27, and an ENB of 5.25; and in the maximum ENCB allocation, the ENB constraint is not even binding, which allows this portfolio to have the highest ENCB (7), together with an ENB of 6 and an ENC of 4.14. The worst drawdown experienced by the maximum ENC portfolio is still lower than without the ENB constraint, at 26.7% as opposed to 38.5%. These results indicate that portfolios that are correctly diversified across factors and across assets tend to better sustain severe bear market conditions, as in the second half of 2008 when multiple asset classes plunged together.

CONCLUSIONS AND PERSPECTIVES

The ability to construct well-diversified portfolios is a challenge of critical importance in multi-asset strategies. A seemingly well-diversified allocation (e.g., equally weighted, minimum variance, maximum Sharpe ratio, or risk parity) to asset classes may well result in a portfolio that is heavily concentrated in terms of underlying factor exposures. In this context, it is important to calculate and manage the effective number of bets, which is a measure of the distribution of factor contributions to risk in a portfolio. But the value of the ENB depends on the set of underlying factors, which are unobservable by nature. For the purpose of identifying these factors, the minimum linear torsion technique has the advantage over standard principal component analysis that it generates easier to interpret factors. In unreported results, we have also been able to verify that MLT factors are much more stable with respect to changes in-sample periods compared to PCA factors.

Given that underlying factor values are estimated rather than observed, they are subject to the usual estimation risk problem, which can effectively be mitigated by imposing weight constraints or a minimum level of diversification across assets. For instance, factor risk parity portfolios, which equate the contributions of factors to volatility, often involve sizable short positions and exhibit high volatility levels, suggesting the need to impose constraints on leverage or volatility. Overall, the ideal situation is to ensure at the same time a proper balance of risk across assets and across factors, so as to take advantage of factor diversification without having to rely solely on estimated implicit factors.

A practically useful extension of the analysis conducted in this article would focus on using these techniques to construct efficient single-asset-class benchmarks, as opposed to multi-asset allocation benchmarks. For example, one may be tempted to investigate the construction of equity portfolios subject to minimum levels of diversification in terms of contributions of the commonly used factors such as value, size, momentum, low volatility, quality, etc. We leave this extension for further research.

APPENDIX A

Formal Derivation of the Number of Factor Risk Parity Portfolios

Consider the representation of factors in terms of asset expected excess returns: $F = A'R$, where A is a nonsingular change-of-base matrix. A portfolio achieves factor risk parity if, and only if, the absolute values of all factor exposures satisfy.

$$\left|\beta_{kp}\right| = \frac{\sigma_p}{\sqrt{N}\sigma_{F,k}}, \quad \text{for all } k=1,\ldots,N \qquad \text{(A-1)}$$

This condition is equivalent to

$$\beta_{kp} = \frac{\varepsilon_k \sigma_p}{\sqrt{N}\sigma_{F,k}} \tag{A-2}$$

where ε_k is −1 or +1.

Asset weights are given by $w = A\beta_p$, and they must satisfy the budget constraint, $w'1 = 1$. Substituting the expression of exposures from Equation A-2 gives the following asset weights:

$$w_i = \frac{1}{\sum_{j,k} \frac{\varepsilon_k}{\sigma_{F,k}} A_{jk}} \sum_k \frac{\varepsilon_k}{\sigma_{F,k}} A_{ik} \tag{A-3}$$

There are 2^N possible choices for the N-tuple $(\varepsilon_1, \ldots, \varepsilon_N)$, but as we see from this expression, the sequences $(\varepsilon_1, \ldots, \varepsilon_N)$ and $(-\varepsilon_1, \ldots, -\varepsilon_N)$ lead to the same asset weights. Hence, the set of weight vectors is spanned by fixing $\varepsilon_1 = 1$ and letting the $N - 1$ other coefficients $\varepsilon_2, \ldots, \varepsilon_N$ take the values −1 and +1. There are 2^{N-1} possible sequences $(\varepsilon_2, \ldots, \varepsilon_N)$, so there are at most 2^{N-1} FRP portfolios.

Appendix B

Maximum Sharpe Ratio Portfolio and Factor Risk Parity

Let Σ and μ be the covariance matrix and the vector of expected excess returns of a set of N risky assets. Factor values are $F = A'R$, where R is the vector of excess returns. Thus, the expected excess returns and the covariance matrix of the factors are

$$\begin{aligned} \mu_F &= A'\mu \\ \Sigma_F &= A'\Sigma A \end{aligned} \tag{B-1}$$

The weights of the maximum Sharpe ratio portfolio are

$$w = \frac{1}{\nu}\Sigma^{-1}\mu \tag{B-2}$$

where ν is the sum of the elements of the vector $\Sigma^{-1}\mu$, so that the weights add up to 1. Thus, the factor exposures are

$$w = \frac{1}{\nu}\Sigma_F^{-1}\mu_F \tag{B-3}$$

Note the similarity between the expressions of asset weights and factor exposures. Because the covariance matrix of factors is diagonal, the contribution of factor k to the portfolio variance, σ_p^2, is

$$c_{F,k} = \frac{\beta_{kp}^2 \sigma_{F,k}^2}{\sigma_p^2} = \frac{1}{\nu^2 \sigma_p^2}\left[\frac{\mu_{F,k}}{\sigma_{F,k}}\right]^2 \tag{B-4}$$

These contributions are equal if, and only if, the absolute Sharpe ratios of the factors, $|\mu_{F,k}|/\sigma_{F,k}$, are all equal.

ENDNOTES

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[1]Amenc and Goltz [2016] summarize the historical evidence and the economic rationale for these effects. A detailed survey of the literature can be found in Amenc et al. [2015].

[2]Ang and Bekaert [2002] solve an international regime-dependent asset allocation model with three equity indices and possibly a risk-free asset.

[3]The sign of asset weights in the definition of a factor is a purely a matter of convention. All signs can be flipped, changing the factor into its negative, without the explanatory power of the factor being impacted.

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