An EDHEC-Risk Institute Working Paper

A Financially Justifiable and Practically Implementable Approach to Coherent Stress Testing

September 2018
Portfolio weights back to their original weights, can be a source of additional performance.
Abstract

We present an approach to stress testing that is both practically implementable and solidly rooted in well-established financial theory. We present our results in a Bayesian-net context, but the approach can be extended to different settings. We show i) how the consistency and continuity conditions are satisfied; ii) how the result of a scenario can be consistently cascaded from a small number of macrofinancial variables to the constituents of a granular portfolio; and iii) how an approximate but robust estimate of the likelihood of a given scenario can be estimated. This is particularly important for regulatory and capital-adequacy applications.
**ABOUT THE AUTHOR**

**Riccardo Rebonato** was previously Global Head of Rates and FX Research at PIMCO. He also served as Head of Front Office Risk Management and Head of Clients Analytics, Global Head of Market Risk and Global Head of Quantitative Research at Royal Bank of Scotland (RBS). Prior joining RBS, he was Head of Complex IR Derivatives Trading and Head of Head of Derivatives Research at Barclays Capital. Riccardo Rebonato has served on the Board of ISDA (2002-2011), and has been on the Board of GARP since 2001. He was a visiting lecturer in Mathematical Finance at Oxford University (2001-2015). He is the author of several books, in particular having published extensively on interest rate modelling, risk management, and most notably books on SABR/LIBOR Market Model pricing of interest rate derivatives, as well as on the use of Bayesian nets for stress testing and asset allocation. He has published articles in international academic journals such as *Quantitative Finance*, the *Journal of Derivatives* and the *Journal of Investment Management*, and has made frequent presentations at academic and practitioner conferences.
1. Introduction and Motivation
In the wake of the 2007-2009 financial crisis a broad consensus has been reached that relying on purely statistical (usually frequentist) techniques to manage financial risk is both unwise and difficult to justify. Stress testing has often been touted as the answer (or, at least, one of the answers) to the shortcomings of managing financial risk using purely statistical tools. Regulators have given an additional reasons to ‘treat stress testing seriously’ through the association between how much capital systemically important financial institutions (SIFIs) should hold and the outcomes of stress-testing exercises.\(^1\) As a result of this, there has recently been a resurgence of academic interest in the area of stress testing. In this paper we make a contribution to this area of research i) by grounding stress testing and scenario analysis in sound financial theory; ii) by suggesting a computationally efficient way to propagate consistently shocks from a small number of prices or macrofinancial variables to the many constituents of a realistic portfolio (a problem that has been referred to in the literature as the dimensionality problem); and iii) by showing how one can ensure that the normal-times joint distribution of risk factors can be consistently recovered from the stressed distribution as the probability of occurrence of the stress event goes to zero. The first two contributions are particularly relevant for (micro-) prudential applications of stress testing and the latter to portfolio-allocation. In this introductory section we describe more precisely the three main contributions of the paper, we explain why they are important, and we briefly describe how our proposed approach fits within the current research programme on stress testing.

A good starting point is the useful distinction made by the ECB (2006) between historical, hypothetical, probabilistic and reverse-engineered scenarios. Broadly speaking, historical scenarios apply to today’s portfolio the moves in market and/or macrofinancial variables that occurred in the past: the post-Lehman events, the bursting of the dot.com bubble or the LTCM crisis are typical examples in this category. Hypothetical scenarios apply to the current portfolio moves in market and/or macrofinancial variables that are expected to occur if some subjectively-chosen scenario were to materialise: the breakup of the EURO currency block, or the outcome of a referendum such as Brexit fall in this category. Probabilistic scenarios apply shocks obtained from a joint distribution of market and/or macrofinancial variables that has been estimated on the basis of historical data; they may but need not employ statistical techniques — such as Extreme Value Theory — designed to probe the tails of the distribution. Finally, reversed-engineered scenarios try to identify the most likely moves in market and/or macrofinancial variables that can give rise to a given maximum acceptable loss (see, eg, Rebonato, 2018 for a recent discussion of this topic).

The distinctions are more blurred that these terse definitions may suggest: ‘objective’ probabilistic approaches still require expert-knowledge-based subjectivity (such as the decision of which data are relevant to the current market conditions, and which trade-off should be struck between relevance and abundance of data); and even the most subjective hypothetical scenarios are necessarily informed by statistical information about past occurrences of ‘similar’ or ‘relevant’ past stress events. Indeed, it has been forcefully argued that the currently unsatisfactory state of stress testing calls for a stronger role of judgment in probabilistic approaches, either in the output (see, eg, Alfaro and Drehmann, 2009) or in the inputs (as in Breuer, Jamdaka, Mencia and Summer, 2009).

Whatever approach is used, the Basel Committee on Banking Supervision (2005) recommends scenarios that are, at the same time, plausible, severe and suggestive of risk-reducing action. It is important to stress that the notion of plausibility can be easily, but unhelpfully, confused with some level of probability mass for a given set of events obtained from a statistical approach (probabilistic scenario): because of its analytical tractability, this is indeed the most commonly taken approach, as in Breuer, Jandaka, Rheinberger and Summer (2009), Breuer, Jandaka, Mencia and Summer (2009), Flood and Florenko (2013), Glasserman, Kang and Kang (2014). However, an event such as, say, the break-up of the Euro currency — with the attending unprecedented changes in dependencies among different variables and, perhaps, the creation of

\(^1\) For a survey of the stress-testing-related regulatory initiatives of the post-Basel-II period, see, eg, Cannata and Quagliariello (2011), Chapter 6 and Part III in particular.

\(^2\) Breuer, Jamdaka, Mencia and Summer (2009) clearly highlight this aspect of model risk — also raised in Berkowitz (2000) — for their approach, which is based on the Mahalanobis distance: “We measure plausibility of scenarios by its Mahalanobis distance from the expectation. This measure of distance involves a fixed distribution of risk factors, which is estimated from historical data.” (page 333)
new variables, such as the 'new drachma' — was very plausible in the Summer of 2012, but simply not present in the data, and therefore impossible to tackle with purely probabilistic models. Probabilistic and hypothetical scenarios therefore complement each other.

Another challenge with stress analysis often mentioned in the literature is what is referred to in this paper as the dimensionality problem: typically, a relatively small number of macro or market variables are shocked (probabilistically or subjectively), but the value of a realistic portfolio depends on granular variables that are often orders of magnitude more numerous. Breuer, Jandaka, Rheinberger and Summer (2009) show that the optimal way to propagate shocks from the few to the many variables is to set the non-directly-shocked factors to their conditional expectation given the value of the stressed factors. The technique we present in Section 11 makes use of this insight in a novel and efficient way.

Stress testing has importance beyond the macro- and microprudential area. Asset managers are keenly interested in integrating in a coherent manner the possibility of severe market distress in their asset allocation procedure. Since diversification is key in this process, and since codependencies between risk factors can dramatically change in periods of financial distress, this poses an obvious challenge. The challenge is compounded by the need, further discussed in what follows, to integrate in a coherent fashion the 'business-as-usual' and the distress portfolio allocations: arguably, as the probability of a distressed event goes to zero, the two allocations should converge. While this condition can be recovered in the case of probabilistic scenarios, it is not easy to ensure that this is the case with hypothetical scenarios. See Rebonato and Denev (2014) for applications of stress testing to portfolio management, and for one possible solution of this problem. In this paper we offer a simpler and financially better-justifiable solution.

These problems (and many more, not mentioned for the sake of brevity) have been widely recognised and amply discussed in the current literature. Perhaps because of the greater emphasis given to probabilistic approaches, much less attention has been given to another important problem that one invariably encounters with hypothetical scenarios, but which arises in a latent form in probabilistic methods as well — a problem whose solution is a key contribution of this paper. It can be stated as follows. When prices of (usually a large number) of assets are moved by large amounts and in 'wild' patterns, it is important to ask whether these configurations of prices are attainable given some basic financial requirements, such as absence of large arbitrage opportunities. We know for instance from financial theory that a relatively small number of factors should and can satisfactorily describe the changes in prices of all assets. These factors can well include liquidity and market sentiment — factors, that is, that may be shocked very strongly, and perhaps to unprecedented extents and in never-before-realised configurations, in a stress situations. The moves in all these factors are in general unconstrained and can be as 'wild' as one may wish. However, the dimensionality reduction (modulo idiosyncratic terms) when moving from the space of the factors to the space of the asset prices imposes serious constraints on the possible moves of prices. These concerns of financial consistency are prominent in term structure modelling, where statistical (VAR-based) and model (no-arbitrage-based) approaches have been used for prediction purposes. Indeed, in the case of fixed-income asset pricing one of the recurring concerns is that the predictions produced by purely statistical models are ignorant of no-arbitrage conditions. But there nothing special about fixed income securities: if we do believe that prices should reflect security-independent compensations for assuming the undiversifiable factor risk, the issue of financial realizability of arbitrarily-assigned price moves remains, particularly in the asset management context.

The potential 'impossibility' of arbitrary price moves is of course very apparent when one deals with hypothetical scenarios. One may think that this should not be the case with probabilistic scenarios, at least to the extent that the underlying joint distribution of price changes has been obtained from historical

3 - See Diebold and Rudebush (2013) for a discussion of the topic, and Joslin, Anh and Singleton (2013) for the conditions under which non-statistical information improves the predicting power of probabilistic models.
market data (data, that is, that at the time of collection must have embodied the financial constraints alluded to above). However, when one deals with severe scenarios, one is typically extrapolating into the tails of joint distributions where actual market occurrences are very rare or perhaps absent altogether. Statistical techniques such as Extreme Value Theory are statistically very well understood, but they know nothing about financial constraints. In this paper we show how these financial consistency constraints can be naturally satisfied.\(^4\)

Summarising, these are the requirements that a useful and theoretically justifiable stress testing programme should satisfy:

1. it should cover ‘severe but plausible’ scenarios (see, eg, BIS (2009));
2. it should reflect events which have not necessarily occurred in the past;
3. it should lend itself to the identification and implementation of hedging or corrective actions;
4. if it has to be used for capital adequacy assessment, at least an order-of-magnitude estimate of the probability of the stress outcome must be provided: ‘thinking the unthinkable’ — as one of the fashionable stress-testing slogans recommends — is not helpful;
5. it must be grounded in financial theory: a subjective element in stress testing is probably inevitable, and arguably to be welcomed; however, an unstructured assignments of shocks and probabilities to risk factors and asset prices is unlikely to satisfy elementary financial and logical requirements of consistency;
6. if it truly has to be used in situation of market distress, a useful stress testing exercise must be updated and executed in almost real time. Unfortunately, the responsiveness of stress testing programme during the 2007–2009 crisis was found seriously wanting. Indeed, during the recent crisis the International Monetary Fund (2010) asked market institutions the pointed question: “If even the most obvious stress-test took many weeks to prepare and assess, how could these tests meaningfully be used to manage risk?”\(^6\)

Of course, in reality they were not;

7. it must handle the ‘dimensionality curse’: both for cognitive and for computational reasons realistic scenarios can only shock a relatively small number (say, order \(O(101)\)) of salient risk factors. However, the realistic portfolios of asset managers are affected by \(O(102) – O(103)\) prices; and systemically important financial institution may have to deal with \(O(105)\) prices. The problem then arises of how one can propagate the scenario-generated shocks from the relatively small number of risk factors stressed ‘by hand’ in the construction of the stress scenario to the multitude of prices that affect the value of a portfolio.

Given this context and the requirement outlined above, this is what our approach can offer.

1. It solidly roots the stress testing exercise in well-established financial (asset-pricing) theory. This can make the approach of particular interest to portfolio managers; however, its solid theoretical grounding constitutes a more generally desirable feature, in that, no matter how severe the scenario considered, the outcome should still be compatible with fundamental conditions of no arbitrage. (See footnote 16 for an elaboration of this point.) When, instead, security prices are more or less arbitrarily shocked following intuition and ‘hunches’, there is no guarantee that the resulting prices should reflect any feasible set of ‘fair’ compensations for the exposures to the underlying risk factors — and, indeed, in naive approaches in general they will not.

2. It allows an approximate estimation of the probability of an assigned scenario. As mentioned above, this is essential for capital adequacy purposes.

3. It naturally suggests corrective actions, thereby answering the concerns in Basel Committee on Banking Supervision (2005).\(^7\)

4. It is built in such a way as to automatically satisfy what we call the consistency and continuity conditions: these reflect the requirement that, as the probability of the stress event goes to zero, the covariances among the risk factors should revert to their ‘business-as-usual’ values, and that they should do so without discontinuities. Again, this condition is essential for portfolio managers, who would naturally want, but usually fail, to see their ‘normal-times’ optimal asset allocations recovered as a limiting case when the probability of a given stress scenario goes to zero.

5. It provides a way to propagate consistently the stress event from the prices of risk factors moved ‘by

---

4 - Purely historical scenarios are immune to this criticism. Useful as they are, their shortcomings are well known.
5 - page 8: “A stress test is commonly described as the evaluation of the financial position of a bank under a severe but plausible scenario to assist in decision making within the bank.”
6 - page 13
7 - The stress-testing approach we propose is built on causal foundations. For a thorough discussion of how a causal (as opposed to associative, as embodied in statistical correlations) organisation of information leads much more directly to intervention, see Pearl (2009) and Pearl and Mackenzie (2018), Chapter 7 in particular.
hand’ to the potentially very large number of securities in a complex portfolio, and it does so in such a way as to satisfy the requirements set in Breuer, Jandaka, Rheinberger and Summer (2009). Doing so allows one to handle the dimensionality curse alluded to above. 6. Once a considerable amount of background preparation work is carried out once and for all, the stress-testing procedure we recommend can then be carried out almost in real time. This takes care of IMF-like concerns about real-time intervention.

It has been argued (Rebonato, 2010, Rebonato and Denev, 2014) that a Bayesian-net approach is best suited to produce coherent scenarios. The practical and conceptual advantages offered by this technique over competing approaches are the ease of interpretation, of sensitivity analysis and of critical interrogation by intelligent but not necessarily mathematically-versed decision makers — and, of course, a solid conceptual and mathematical basis. For this reason, for the sake of concreteness in what follows we will cast the discussion in the language of Bayesian nets. However, we stress that the solutions we offer are applicable to any stress testing approach in which the shocks to a number of macro factors and the associated probabilities can be assigned, and must be propagated to the portfolio prices. Needless to say, assigning the joint probabilities of these complex shocks in a logically coherent manner is not easy, and this is why we recommend the use of Bayesian nets. We remain however agnostic in the treatment that follows as to how the shocks and joint probabilities have been obtained.

For the sake of clarity, it is important to clarify which aspects of stress testing our work does not cover. First and foremost, the choice of stress event is very important in the probabilistic, hypothetical and reverse-engineering modes. Many of the paper in the current literature address exactly this problem (see, eg, Glasserman, Kang and Kang (2014), Flood and Korenko (2013) for probabilistic approaches and Golub, Greenberg and Ratcliffe (2018) for the hypothetical case). In our approach we do not address this problem, and simply assume that one or more hypothetical scenarios have been chosen using any of these methods.

Second, the economic grounding of the practice of stress testing is of great theoretical relevance (Parlatore and Philippon, 2018, for instance, deal with the learning and design part of the process whereby a risk-averse agent – such as a regulator — may seek to learn about the exposures of agents – banks — to a number of risk factors; Corbae, D’Erasmo, Galaasen, Irrazabal and Siemsen, 2017, develop a structural model for microprudential stress testing in a partial equilibrium setting in which the decision to exit positions are endogenous). Foundationally important as they, we do not deal with these problems.

Finally, the subjective inputs that enter any hypothetical scenario (such as the choice of subjective conditional probabilities of events) should be ‘cognizant about’ and consistent with the corresponding frequentist or marketimplied quantities. Also this important topic is beyond the scope of our study.

---

8 - For instance, in markets where options are traded a continuum of strikes gives access to the state price densities (in the pricing measure). See, Breeden and Litzenberger (1978). Market-implied volatilities of options have been shown to be quickly responsive to changing market conditions (but they also include a component due to the changing market price of volatility risk). There is very little information, however, about market-implied correlations.
2. Bayesian Nets for Stress Testing
Rebonato (2010), Rebonato and Denev, (2010, 2011), Denev (2015), Rebonato and Denev (2014) and Rebonato (2017) have argued in detail that the Bayesian net technology lends itself particularly well to stress-testing applications. As this technique lends itself particularly well to the treatment at hand, in this section we give a brief account of the approach so as to make the treatment self-contained, and to clarify some ideas and the terminology.\(^9\) We stress again, however, that the approach we present in this paper does not logically rely on Bayesian nets. Once the shocks to a number of macro variables (described below) and the attending probabilities are assigned (in whatever way) our procedure remains valid without further reference to where the shocks 'come from'. In the classification outlined in the Introduction, the procedure is 'hybrid', in that it uses hypothetical inputs to obtain a (subjective) probability distribution of risk factors (and prices). Albeit in a different way, the Bayesian net procedure heeds the recommendations by Breuer, Jamdaka, Mencia and Summer, 2009 that subjectivity and judgement calls should enter in the input phase (rather than in the output, as Alfaro and Drehmann, 2009 recommend).

Bayesian nets can be formally described as directed acyclical graphs equipped with conditional probability tables. They are directed, because they establish a parent/child relationship (signified by directed arrows) between nodes (variables) in the net. They are acyclical, because the arrows cannot form a closed path.

The variables in the net can either be Boolean variables (in which case they assume TRUE / FALSE values); or they have a finite (and, in practice, small) number of possible realisations. When a directed arrow points from variable \(A\) to variable \(B\), we say that \(A\) is the parent and \(B\) its child. So, for instance, in Fig (1), the variable \(C\) has variables \(A\) and \(B\) as parents.

The topology of the net (ie, the way the nodes are arranged and connected by the directed arrows) gives information about which variables are conditionally independent. Conditional independence is much weaker (and much more common in reality) than independence, and arises when, given random variables \(A, B\) and \(C\), probabilistic statements about the variable \(A\) given knowledge of the realisations of variables \(B\) and \(C\) are the same as if only the realisation of variable \(B\) were available:

\[
p(A|BC) = p(A|B). \tag{1}
\]

The full probabilistic information about the variables that occur in a Bayesian net is, of course, embodied in their joint-probability table. Bayesian nets exploit conditional independence to achieve a factorisation of the joint probability, \(p(x_1, x_2, ..., x_n)\), as a product of marginal and conditional probabilities conditioned only on the states of the parents of a node:

\[
p(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} p(x_i|\text{par}[x_i]) \tag{2}
\]

In this expression, \(\text{par}[x_i]\) denotes the parents of variable \(x_i\), and the conditional probability of the root (ie, of the variable that has no parents) coincides with its marginal.\(^10\) It is this factorisation that allows that great computational (and cognitive!) savings afforded by Bayesian nets.

To provide a concrete, if unrealistically simple, example, consider the event 'Central Bank Rises Target Rate', associated in Fig (1) with variable \(D\). In this simplified scenario, the probability of this event is assumed to depend only on whether inflation expectations have increased or decreased. The 'Inflation Expectations' variable (variable \(C\)) in turns depend on the possible realisations of 'Strength of the Currency' (variable \(A\), which can assume values STRONG or WEAK) and of 'Output Gap' (variable \(B\), which can be in state HIGH or LOW). See Fig (1).

---

9 - For a detailed treatment of Bayesian nets, see, eg, Jensen and Nielsen (2007) and Pearl (2009) for the advantages of causally-based Bayesian nets.

10 - The second cornerstone of Bayesian Net construction is the No-Constraint theorem, which says that if one deals only with 'canonical' conditional probabilities — ie, exactly the conditional probabilities required by the conditional probability tables — one can assign to them any number between 0 and 1, and rest assured that the resulting Bayesian Net will be mathematically consistent. See Rebonato and Denev (2013) for a proof of the theorem, and for the definition of canonical conditional probabilities. See Moskowitz and Rebonato and Kwiatkowski for the problems that can arise if one assigns non-canonical conditional probabilities.
The crucial insight is that, if we are happy to assume that the probability of the Central Bank rising rates only depends on the inflation expectations (irrespective of whether these are due to a weakening/strengthening currency or to a sluggish/exuberant economic growth), then the conditional probability of $D$ given $A$, $B$ and $C$ is just equal to the conditional probability of $D$ given $C$:

$$p(D|A,B,C) = p(D|C). \quad (3)$$

In Bayesian-net parlance, the inflation variable 'screens' the variable 'Central Bank Rises Target Rate' from the 'upstream' variables, and it is the systematic exploitation of this property that allows the computational and cognitive savings afforded by Bayesian nets.

Since the theory of how Bayesian Net can be used to model stress events has been covered elsewhere in detail, we do not pursue this aspect any further. All we need for the treatment to follow are i) shocks to macro factors (defined below) and ii) the attending probabilities (howsoever obtained). We assume that these quantities have been obtained (via a Bayesian net or otherwise), and we therefore move to defining precisely what we mean by a stress scenario.
3. What Constitutes a Stress Scenario
In our approach, a stress scenario reflects a view on the future joint distribution of risk factors that differs from the consensus ‘market view’. Since market views are only rarely and imperfectly accessible, we assume that all investors observe the same time series of past prices and risk factors and the current market conditions, and, based on this information, form a conditional view about the joint distribution of the factors or prices.

The scenario view may be different from the consensus market view for prudential reasons (the probability or severity of adverse states of the world may be increased to explore the resilience of a portfolio); or because the trader or portfolio manager has subjective views which differ from the consensus ‘market view’.

We stress that the consensus ‘market view’ need not reflect a quiet-market condition, or an unconditional distribution of asset returns. Investors, for instance, may well estimate volatilities using conditional techniques such as GARCH-family models. When they do so, in periods of market distress they may well assign higher-than-normal probability to very adverse market moves. In these distressed conditions, a stress scenario view can still differ from the consensus one in being more or less ‘optimistic’ than the conditional view.
4. The Goal
The goal of the Bayesian-net construction is to produce a joint probability distribution for a set of representative market indices (defined below) of interest for a given portfolio in the presence of a stress scenario view.

If the stress outcome for a single representative market index is required, then the probability distribution is univariate; otherwise multi-variate.

From the distribution the expected value of the representative market index or indices can be obtained, as well as their (co)-variances, risk statistics (VaR, CES, etc); etc.

How to propagate the stress shock from the representative to the granular market indices is discussed in Section 11.
5. Definition of the Variables
Several types of variables appear in the set-up we describe. Some (such as the root events and the connecting nodes) are closely linked to a Bayesian net description of the scenario; others occur in virtually all stress-testing applications.

1. Root Events
(equipped with marginal distributions). These are the prime causes or triggers of a stress event.
Examples of root events are unlikely but possible events whose occurrence can strongly affect a set of asset prices. For instance, the break-up of the Euro, or the withdrawal of one country from the European Union are examples of such stress events. Of course, the market may well assign a non-zero probability of occurrence to these events. In the light of the definition in Section 3, a stress event should therefore be interpreted as the difference in the joint probability distribution from the base-case (market) reference level.

2. Connecting Nodes
(equipped with conditional probability tables). These are the transmission channels that 'link' the roots to the leaves of the net.
Example of transmission channels may be the actions of a central bank as a response to the events in the roots; the failure of a major financial institution; the resumption of a programme of Quantitative Easing; etc.

3. Macrofinancial Variables
('leaves' linked to connecting nodes and, possibly, to roots).
These are changes in macroeconomic variables affected, either directly or through the connecting nodes, by the root variables. As explained below, in our approach changes in the macroeconomic variables are assumed to fully determine the expectations of changes in asset prices. Typical macroeconomic variables could be: expected GDP growth, expected inflation, unemployment, market ‘sentiment’ (as defined, for instance in Burnmeister, Roll and Ross (1994)); etc;

4. Representative Market Indices.
These are the prices or values of benchmark market prices or rates, such as, for instance, the 10-year US Treasury Bond yield, the level of the S&P 500 index, the $/Yen exchange rate, the price of oil, etc.
As discussed below, the assumption is made that expected changes in these benchmark prices and rates are fully specified once the changes in the macro variables are given.

5. Granular Market Indices.
These are the prices or values of nonbenchmark market prices or rates affecting the value of a portfolio or trade.
Examples of granular market prices are the prices of, say, the 3- or 7-or 23-year US Treasury bond, the level of the AIM index, the price of an individual stock, etc.
As discussed below, their realisations are assumed only to depend on the realisation of the associated representative market indices. Once the changes in the granular market indices are given, the change in value in the whole portfolio can be calculated.

Given the causal links in a given stress scenario, some macro factors may be unconnected to the roots (ie, in some scenarios no paths may link through connecting nodes the root(s) with a subset of macro factors). In this case, the macro factors in the unaffected subset are all assumed to be in their ‘normal’ state.
6. Notation
We denote each variable associated with macrofinancial factors by an upper case Greek letter, and their realisations by the associated lower-case Greek letter. Variables associated with representative market indices are denoted by an upper case Latin letter, and their realisations by the associated lowercase Latin letter.

Roots and nodes in the Bayesian net that are ‘upstream’ of the macro variables are denoted by $\text{Root}^k$ or $\text{Node}^k$ and their realisations by $\text{root}^k_j$ or $\text{node}^k_j$, respectively (where $j$ denotes the state and $k$ the node or root).

The variable realisations are identified by a subscript, with the subscript 0 reserved for the normal state.

The notation $p(a_j)$ denotes the probability of variable $A$ being in state $k$, and similarly for macro factors.

So, for instance, the notation $\Omega = \omega_0$ and $\text{Node}^k = \text{node}^k_1$ indicates that the macrofinancial variable $\Omega$ is in the normal state and that the $k$-node is in the first state.
7. The Consistency and Continuity Requirements
If all the roots are in the normal state, then, for every root, $\text{Root}^k$, $p (\text{Root}_0^k) = 1$, $p (\text{Root}_j^k = 0), j \neq 0$. When this occurs, we say that the probability of the stress event is zero and that we are in the ‘normal’ state (in the sense of Section 3).

We require that, when the probability of the stress event is zero, the (joint) distribution of the representative market index or indices should be equal to the (joint) distribution of representative market indices that applies in ‘normal’ times. (This is the Consistency Requirement.)

We also require that, as the probability of the stress event goes to zero, the (joint) distribution of the representative market index or indices should converge continuously to the (joint) distribution that applies in ‘normal’ times. (This is the Continuity Requirement.)

As mentioned in the Introduction, these two requirements are particularly important in portfolio-construction applications.
8. Assumptions
We make the following assumptions about the links between macro factors and the representative market indices. The assumptions reflect the definition of normal state given above.

1. We assume that each variable can exist in at least two and at most three distinct states. One of these states must be the normal state.\(^\text{12}\)

2. If all the parents of a given node are in the normal state, then the child node(s) can only be in the normal state.

3. We assume that the Arbitrage Pricing Theory (Ross, 1976) provides an acceptable description of the returns of the representative market indices, given a chosen set of macro factors.

4. As in most common implementations of the Arbitrage Pricing Theory, we assume that to each macro factor one can associate a factor-mimicking portfolio. (This assumption is relaxed in Section 11).

5. We assume that the ‘market prices of risk’ (i.e., the compensations per unit risk) derived by the application of the Arbitrage Pricing Theory procedure do not change with the stress event. So, the shocks associated with a given stress scenario can be as large as desired, but the associated compensations per unit of risk are not state-dependent. We note that this assumption could be relaxed, as empirical information has recently become available about the state-dependence of risk premia, at least in some important asset classes (see, e.g., Cochrane and Piazzesi (2005), Cieslak and Povala (2010)). We work in the constant-market-price-of-risk setting for the sake of clarity and simplicity.

\(^{12}\) - Theoretically, we could accommodate as many states as required; the limit of three is imposed for pragmatic reasons.
To keep notation light and help intuition, consider the case of just two macro variables, $\Omega$ and $\Gamma$, that can exist in states $\{\omega_0, \omega_1\}$ and $\{\gamma_0, \gamma_1\}$. Let
\[
p(\omega_0, \gamma_0) = p^{00} \\
p(\omega_0, \gamma_1) = p^{01} \\
p(\omega_1, \gamma_0) = p^{10} \\
p(\omega_1, \gamma_1) = p^{11}
\] (4)
be the joint probabilities produced (for instance, by the Bayesian net) for four possible combinations of values of the macro variables, $\Omega$ and $\Gamma$.

We show first how the normal state can be described (Section 9.1), and then we extend to the case of stressed states (Section 9.2).

### 9.1 The Normal Case

Let $f^0_\Omega(t)$ and $f^0_\Gamma(t)$ be the zero-mean self-financing factor-mimicking portfolios associated with the macro variables $\Omega$ and $\Gamma$.\(^ {13}\)

Typical macro factors can be, for instance, a GDP growth factor, an inflation factor, a confidence/liquidity factor (see, e.g., Burmeister, Roll, and Ross (1994) for a discussion). The associated factor-mimicking portfolios would then be self-financing portfolios with long (short) exposures to assets that benefit (perform badly) when the macro factor is in a high (low) state.

We note that the CAPM market factor is usually taken to be one such factor, and the self-financing portfolio can be thought of as a long position in the market portfolio financed at the riskless rate (say, with Treasury Bills).

From the Arbitrage Pricing Theory (Ross, 1976) we can then write for the return, $r_i(t)$, of the $i$th representative market index in our two-macro-factor universe
\[
r_i(t) = E[r_i] + \beta_\Omega f^0_\Omega(t) + \beta_\Gamma f^0_\Gamma(t) + \epsilon_i(t)
\] (5)
with
\[
E[f^0_\Omega(t)] = E[f^0_\Gamma(t)] = 0
\] (6)
and $\epsilon_i(t)$ uncorrelated with all the factor-mimicking portfolios:
\[
cov[\epsilon_i(t), f^0_j(t)] = 0, \forall i, j .
\] (7)
We also have (lack of serial correlation)
\[
cov[f^0_j(t), f^0_j(t')] = 0 .
\] (8)

We stress, however, that the 'noise' terms $\epsilon_i(t)$ need not be uncorrelated across representative market indices.

The 'betas', $\beta_\Omega$ and $\beta_\Gamma$, in Equation (5) are obtained following the usual procedure (see, e.g., Fama and Macbeth (1973)) through a first-stage timeseries regression, and represent the 'loadings' of return $i$ on the two factors (as proxied by their factor-mimicking portfolios). Each loading is given by
\[
\beta_{i\Pi} = \frac{\sigma_i}{\sigma_{\Pi}} \rho_{i\Pi} ,
\] (9)
where $\Pi$ signifies the generic macro factor, $\sigma_i$ the return volatility for market index $i$, and $\sigma_{\Pi}$ the volatility of factor-mimicking portfolio $\Pi$.

We stress that the factor-mimicking portfolios have an expectation of zero (which can be always achieved by de-meaning). Therefore, individually they represent the variability of each factor, and jointly they carry information about their co-dependence.

Still following the standard procedure, we then carry out a second-phase regression (now cross-sectional), in which the 'betas', $\beta_\Omega$ and $\beta_\Gamma$, now play the role of explanatory (right-hand) variables:
\[
E[r_i] = \lambda_0 + \lambda_\Omega \beta_\Omega + \lambda_\Gamma \beta_\Gamma + \eta_i .
\] (10)

From this equation, using standard arguments (see, e.g., Cochrane (2001)) one obtains that $\lambda_0 = r_f$, and that $\lambda_\Omega$ and $\lambda_\Gamma$ are the 'market prices of risk' associated with the macro variables (risk-mimicking portfolios) $\Omega$ and $\Gamma$, respectively.

By substitution one finally obtains\(^ {14}\)
\[
r_i(t) = r_f + \beta_\Omega [\lambda_\Omega + f^0_\Omega(t)] + \beta_\Gamma [\lambda_\Gamma + f^0_\Gamma(t)] + \epsilon_i(t) .
\] (11)

\(^ {13}\) For a practitioner-friendly introduction to the Arbitrage Pricing Theory and the construction of factor-mimicking portfolios, see Burmeister, Roll, and Ross (1994).

\(^ {14}\) The exact equality sign in Equation (10) applies in the case of well-diversified factor portfolios. We assume that this is always the case.
With some modifications presented in Section 9.3, this is the fundamental equation that we are going to use for our stress testing application.

Summarising,

- the zero-mean factor-mimicking portfolios, \( f_i^0 (t) \), provide the systematic variability to which the representative market indices are exposed;
- the 'betas', \( \beta_i \), provide the loadings of each representative market index onto the factor-mimicking portfolios;
- the 'market prices of risk', \( \lambda_i \), provide the representative-market-index independent compensations for bearing a unit of risk in the portfolio (factor), \( \Pi_i \);
- the terms \( \epsilon_i (t) \) provide idiosyncratic (non-priced) variability associated with each representative market index.

This set-up describes the 'normal state' set of returns.

### 9.2 The Non-Normal States

In our set-up a shock to a macro variable is obtained by giving a statedependent shock away from zero to the expectation of the associated factor-mimicking portfolio.\(^{15}\) More precisely, we proceed as follows.

Suppose that at least one of the roots is not in the normal state. Given our assumptions, this implies that, in general, at least one macro variable, \( \Omega \), will be in a state, call it \( \omega_1 \), other than the normal state, \( \omega_0 \).

In our set-up we describe this condition by imposing that in this state of the world the ith factor-mimicking portfolio should no longer be described by a random variable with mean zero and variance \( \sigma_i^2 \). Instead, if state \( \omega_1 \) prevails we centre the shocked factor-mimicking portfolio a desired number of standard deviations above or below 0. The shock can be made more or less severe by relocating the first moment of the distribution of the factor-mimicking portfolio a greater or a smaller number of standard deviation away from zero. Since the distribution of the portfolios is assumed to be known, we can associate, at least approximately, a probability to each of the shocks thus defined.

For the two-macro-factor case discussed above, if we still assume for simplicity that both macro factors can only exist in a normal and in a 'bad' state (the latter labelled by the index '1'), the values assumed by the return on the ith representative market index, \( r_i(t) \), conditional on the possible combinations of values of the factor-mimicking portfolios, are then given by

\[
\begin{align*}
  r_i^{00} &= r_f + \beta_{i\Omega} \left[ \lambda_\Omega + f_{i\Omega}^0 (t) \right] + \beta_{i\Gamma} \left[ \lambda_\Gamma + f_{i\Gamma}^0 (t) \right] + \epsilon_i (t) \\
  r_i^{01} &= r_f + \beta_{i\Omega} \left[ \lambda_\Omega + f_{i\Omega}^0 (t) \right] + \beta_{i\Gamma} \left[ \lambda_\Gamma + f_{i\Gamma}^1 (t) \right] + \epsilon_i (t) \\
  r_i^{10} &= r_f + \beta_{i\Omega} \left[ \lambda_\Omega + f_{i\Omega}^1 (t) \right] + \beta_{i\Gamma} \left[ \lambda_\Gamma + f_{i\Gamma}^0 (t) \right] + \epsilon_i (t) \\
  r_i^{11} &= r_f + \beta_{i\Omega} \left[ \lambda_\Omega + f_{i\Omega}^1 (t) \right] + \beta_{i\Gamma} \left[ \lambda_\Gamma + f_{i\Gamma}^1 (t) \right] + \epsilon_i (t).
\end{align*}
\]  

Each of these possible realisations occurs with probabilities \( p^{00}, p^{01}, p^{10} \) and \( p^{11} \), respectively, where each \( p^{ij} \) is the joint probability of the first macro factor being in state \( i \) and the second in state \( j \).

Therefore in the presence of a stress event the return on the representative market index, \( r_i(t) \), is given by

\[
  r_i = r_i^{00} p^{00} + r_i^{01} p^{01} + r_i^{10} p^{10} + r_i^{11} p^{11} + \epsilon_i (t).
\]  

Given this representation, one can calculate the expected value of the return for the ith representative market index, \( r_i(t) \). Let

\[
  E \left[ f_{i\Omega}^1 (t) \right] = \omega_1
\]

and

\[
  E \left[ f_{i\Gamma}^1 (t) \right] = \gamma_1.
\]

So, for instance, \( \omega_1 \) and \( \gamma_1 \) could represent a 2- and 3-standard deviation shock to inflation and GDP growth expectations, respectively.

\(^{15}\) If we so wished, we could also change its volatility. We do not pursue this angle, but the generalisation presents no conceptual difficulties.
Then, remembering that $E \left[ \int_0^t f_i(\tau) \right] = 0$, we have

$$E \left[ r_f \right] = r_f + (\beta_{i\Omega} \lambda_{i\Omega} + \beta_{i\Gamma} \lambda_{i\Gamma}) p^{00} + \beta_{i\Omega} \lambda_{i\Omega} + \beta_{i\Gamma} [\lambda_{i\Gamma} + \gamma_i] p^{01} + (\beta_{i\Omega} [\lambda_{i\Omega} + \omega_i] + \beta_{i\Gamma} \lambda_{i\Gamma}) p^{10} + (\beta_{i\Omega} [\lambda_{i\Omega} + \omega_i] + \beta_{i\Gamma} [\lambda_{i\Gamma} + \gamma_i]) p^{11}. \quad (16)$$

with the probabilities $p^{ij}$ determined by the structure of the Bayesian net (or otherwise if a Bayesian-net approach is not used). Higher moments can, of course, be computed in a similar manner.

It is important to note that, if it so happens that the stress event is ‘not active’ (i.e., if we are in the normal state), then $p^{00} = 1$, $p^{01} = p^{10} = p^{11} = 0$, and we therefore recover by construction the correlation and variance structure in the normal state. As all the quantities that multiply the conditional probabilities $p^{ij}$ are obviously finite, we are assured of continuity. The consistency and continuity requirements of Section 7 are therefore automatically satisfied.

In sum:

- We have presented a way to propagate (possibly through a Bayesian net) the effects of a root stress event to a number of representative market indices.
- We have done so in such a way that (within the assumptions made) the effects of the shocks in the macro factors (the factor-mimicking portfolios) are propagated to the representative market indices in an arbitrage-free manner (this is a result of our adopting the normative framework of the APT).
- Since the covariance structure of the factor-mimicking portfolios is known, we can estimate the probability of any combination of shocks to the factor-mimicking portfolios (or conversely, for any set of joint probabilities obtained in the Bayesian net construction, we can work out the magnitude of the associated factor shocks). While a degree of subjectivity obviously unavoidably remains in the construction of the scenario (and, arguably, this is desirable), this subjectivity is now structured and ‘controlled’.

- By construction, we have ensured that, as the probability of the stress event goes to zero, the covariance structure of the representative market indices is exactly recovered. Therefore the business-as-usual asset allocation is in the limit recovered.

It remains to be shown how the shocks to the representative market indices can be propagated to the granular market indices. Before doing so, however, we highlight some of the weaknesses and shortcomings of the proposed approach.
10. Limitations and Weaknesses of the Approach
As any modelling approach, the one we propose has some drawbacks. We think that they are blemishes and not fatal flaws, but we prefer to discuss them at this stage, rather than tucking them away in the last paragraphs of the paper.

To begin with, the theoretical bedrock upon which our approach rests is the Arbitrage Pricing Theory. This has been shown to provide a more satisfactory description of cross-sectional variations in asset returns than the CAPM theory; however, it is far from being a perfectly satisfactory theory. For our applications, the main desiderata from an asset pricing model is its ability to propagate shocks from factors to prices in a manner which is logically consistent and which does not do violence to no-arbitrage.\footnote{One could argue that during a severe stress the actions of the arbitrageurs may be seriously impeded, and that therefore imposing no-arbitrage in a stress condition is either unnecessary or, perhaps, even counterproductive if the stress test is used for prudential purposes. We believe that even in distressed market conditions no-arbitrage remains a useful reference point, and that temporary deviations from perfect absence of arbitrage can be modelled by adding uncorrelated shocks, $E_i(t)$.
}

As presented, our approach makes use of factor-mimicking portfolios. Their creation is a well explored task in empirical asset pricing studies, but it remains not-trivial. It is possible that, whatever label one may attach to a self-financing factor portfolio, this may only imperfectly reflect the macro variable one intended to mimic. A poor match would mean that one would assign a shock to a portfolio believing that it is proxying for a given macro factor, while it actually also reflects a host of other uncontrolled factors. The problems associated with the creation of factor-mimicking portfolios can be circumvented by working directly with de-meaned and standardised macrofinancial factors (as originally presented in Chen, Roll and Ross (1986) and more recently recommended, for instance, in Amenc and Le Sourd (2003).) By doing so, some of the theoretical links between the APT model and investors’ preferences become more difficult to recover\footnote{Since portfolio payoffs, not factors, covary with consumption, and risk premia come from the covariance between a payoff and consumption, using portfolio allows a more direct interpretation of the risk premia. However, as long as the mapping between the factors and the associated portfolios is linear (as it is invariably the case), a translation from one set of ‘coordinates’ to the other can always be accomplished.}. However, for the application at hand, the direct use of factors instead of their associated portfolios can still provide a solid theoretical foundation to scenario construction.

The strongest assumption is probably that the regression ‘slopes’, $\lambda_J$ in the cross-sectional regression (10) (ie, the compensations per unit risk) should be state-independent. So, the shocks to the macros factors may be very ‘wild’ and different from anything which has occurred in the past, but, given these possibly very atypical joint shocks, the compensation for bearing the associated risks remains constant. As mentioned, there is evidence that, at least in the fixed-income world, this is not the case. One could in principle envisage embedding information about the state-dependence of market prices of risk. We leave this for future developments.

Finally, the procedure may become unsatisfactory to the extent that the chosen factor-mimicking portfolios (or the underlying factors in the simpler implementation alluded to above) significantly failed to span the range of changes in representative market indices encountered in reality. In an attempt to fix this, one can, of course, always assign additional exogenous shocks to the idiosyncratic terms, $E_i(t)$, but these additional shocks would now have a distinctly ad hoc flavour. In empirical applications, we have not found the incomplete spanning to be a concern.
11. From Representative to Granular Market Indices
We have left unresolved the problem of how to propagate the shocks from the representative of the associated granular market indices — ie, from, say, the 10-year Treasury yield to the whole Treasury yield curve. As it happens, this is the easiest task, at least if the following assumption is accepted.

**Assumption.** It is assumed that, for a given representative market index, the correlation structure among the associated granular market indices is the same in the stressed and in the normal condition.

If we accept this assumption\(^\text{18}\), in a recent paper Saroka and Rebonato (2015) show how a large number of correlated variables can be 'optimally' moved given exogenous shocks to a much smaller number (say, one or two) of these variables. The purpose of their original paper was how to deform a yield curve given the exogenous views of a portfolio or risk manager. Rebonato (2018) has applied the same approach to reverse-stress-testing and Dupont (2017) and Gaissendress (2017) to the valuation of mid-curve options. As for the 'optimality' referred to above, Rebonato (2018) shows that the resulting deformation has some very desirable properties, in the sense of being a maximum-entropy solution to the conditional deformation problem.

In the present context,
- the large number of correlated variables alluded to above are the granular market indices (say, the many yields that describe the Treasury yield curve) associated with the representative market index or indices shocked using Equation (16) (say, the 10- and 30-year Treasury yields);
- the ‘few’ variables are the very-small-number of representative market indices associated with the same curve (in this example, the 10-year and 30-year yields); and
- the ‘views’ are the shocks to the representative market indices obtained by the procedure described in Section 9.

### 11.1 Illustration of the Procedure
To illustrate the procedure, we refer to the case where the representative market index is a (small number of) benchmark yield(s) of a sovereign Government curve. Then let \(y_i, i = 1, 2, \ldots, N\), be the granular indices that describe \(N\) discrete points of a yield curve, and let \(\Sigma\) be the covariance matrix of changes in the granular yields. After orthogonalising \(\Sigma\) as in

\[
\Sigma = U \Delta U^T
\]

the principal components, \(\Delta x\), are given by

\[
\Delta x = U^T \Delta y = U^{-1} \Delta y
\]

and we have

\[
\Delta y = U \Delta x.
\]

Following Saroka and Rebonato (2015) we now define normalised principal components, \(\Delta p\), by

\[
\Delta p = V^{-1} \Delta y
\]

with

\[
V = U \Lambda^{\frac{1}{2}}.
\]

Then we have

\[
\Delta y = V \Delta p
\]

and

\[
\Delta p = \Lambda^{-\frac{1}{2}} \Delta x,
\]

showing that \(\Delta p\) are indeed volatility-normalised principal components. Then, for any yield \(i\), we have

\[
\Delta y_i = \sum_{k=1,NPC} V_{ik} \Delta p_k.
\]

Now, let us obtain, through the procedure described in Section 9, \(\delta \tilde{y}_i, i = 1, 2, \ldots, n, n << N\), the shocks to the \(n\) representative market yields. Our goal is to generate the most statistically likely move of the whole yield curve that is consistent with all the shocks (‘views’), \(\delta \tilde{y}_i\), associated with a particular configuration of the macro factors (in practice, of the factormimicking portfolios)

If we make some natural distributional assumptions about the joint distribution of the principal components
(see Saroka and Rebonato (2014) and Rebonato (2017)), this can be shown to be equivalent to minimising $S$,

$$\min S = \sum_{k=1}^{N} \delta p_k^2$$

subject to

$$\delta y_i = \sum_{k=1}^{n} v_{ik} \delta p_k,$$

with $m \geq n$ is the number of principal components that we are going to use to deform the yield curve, and $v_{ik}$ are the elements of $V$. If we accept that minimising the sum of squares, $S$, is a reasonable thing to do (see Saroka and Rebonato (2014) and Rebonato (2018) for a justification), then, after defining the Lagrangian, $\mathcal{L}(\delta p, \lambda)$,

$$\mathcal{L}(\delta p, \lambda) = -\sum_{k=1}^{N} \delta p_k^2$$

$$+ \sum_{i=1}^{m} \lambda_i \left( \sum_{k=1}^{n} v_{ik} \delta p_k - \delta y_i \right)$$

and taking the necessary derivatives, one obtains:

$$\frac{\partial \mathcal{L}}{\partial \delta p_j} = -2\delta p_j + \sum_{i=1}^{m} \lambda_i v_{ij},$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = \sum_{k=1}^{n} v_{ik} \delta p_k - \delta y_i$$

As the expressions above are just linear combinations of $\delta p_j$ and $\lambda_i$, solving the zero-derivative condition is simply equivalent to solving the linear system

$$Aw = b$$

with

$$w = (\delta p_1, \delta p_2, \ldots, \delta p_N, \lambda_1, \lambda_2, \ldots, \lambda_m)^T$$

$$b = (0, 0, \ldots, 0, \delta y_1, \delta y_2, \ldots, \delta y_m)^T$$

and

$$A = \begin{bmatrix} -2I_{N \times N} & V_{N \times m} \\ (V_{N \times m})^T & 0_{m \times m} \end{bmatrix}$$

Solving for $w$ gives

$$w = A^{-1}b$$

and this determines the $\{\delta p_i\}$. From this the updated discrete yield curve, $Y$, can be obtained as

$$Y = V \rho.$$

11.2 Empirical Effectiveness of the Procedure

As described, the procedure proposed so far is intuitively appealing, and simple to implement. It is natural to ask whether it does work well in practice. We can provide a flavour of the empirical tests that can be carried out to assess its effectiveness as follows.

First, we start from a market yield curve — eg, the US$ swap curve. We then look at the realisation of the chosen curve one month later. We record the realised change in the 10-year and 30-year rates. These realised changes, shown as green asterisks labelled ‘predictions’ in Fig (2), are our ‘views’.

To test the goodness of the procedure, we compare our ‘guess’ with what actually happened (ie, the actual realised yield curve one month later). This is what is shown in Fig (3) for one particular date.

The two ‘views’ are, of course, recovered perfectly (by construction). But the interesting observation that the whole yield curve is very well recovered. (Note how close the realised and predicted yield curves are.)
A systematic empirical study of the effectiveness of the ‘views’ methodology is beyond the scope of this paper (see however Rebonato (2018) for a detailed empirical analysis), but the results obtained for the date shown is typical, and we found similarly good agreement for all the dates and asset classes that we have examined.
12. Some Practical Examples
In this section we show two simple implementations of the ideas described in the paper, one in a single-transaction context, and one for a multi-currency portfolio problem. Treating these cases in detail would require a paper as long as the present, and we therefore simply provide concrete indications of some salient features.

12.1 Transaction Applications

We call transactional all those applications where the number of assets shocked is smaller than or equal to the number of factors. In this case the factor structure is not strictly necessary because there is always one configuration of factors that can give rise to the chosen prices shocks (however wild). Despite the fact that in this case the factor approach is not required, it can still provide guidance in constructing the scenario.

The transaction example we discuss in this application looks at what the effect of three further rate hikes by Fed by year end would be on a duration and a yield-curve slope trade in US Treasuries. The narrative associated with the scenario is as follows. See Fig (4).

First of all, it is important to distinguish the case when the three hikes in rates are due to strong GDP growth or high inflation (or, of course, both). This is the case because the reason for the increase in rate will influence the conditional probability of the downstream nodes.

Another variable of interest is the change in level of the S&P500 level. This variable has been chosen because of the policy of the Fed during the Greenspan/Bernanke/Yellen chairmanships of cutting aggressively rates to counter financial shocks (the ‘Greenspan put’). The likelihood of a weak or strong performance by the S&P500 clearly depends on whether the three rate hikes have been made, but also on whether the rates hikes were made because of particularly strong GDP growth, or because of strong inflation. Therefore the variable "SP 500" in the net does not screen the variable "3 Rises".

The variables considered so far affect the market liquidity, the expected inflation after the Fed intervention (has inflation been tamed?) and the expected GDP growth after the Fed actions. These

Figure 4: The transactional Bayesian net associated with the scenario "3 rate hikes by the Fed by year end". The numbers next to the various states indicate the probability of occurrence given the conditional probabilities (not shown) and the structure of the net.
three macrofinancial variables are then linked to the two transactions of interest: a duration trade (whereby the investor gains $1,000,000 for every basis point in a parallel increase in the yield curve); and a ‘slope trade’ (whereby the investor gains $2,000,000 for every basis point in yield curve steepening — with steepness defined as the difference between the 10-year and the 2-year yield). Shocks to the level and slope of the yield curve were assumed to be drawn from Gaussian distributions centred on +50 and -20 basis points (with standard deviations of 25 basis points) for the changes in level and on +20 and -15 basis points (with standard deviations of 20 and 10 basis points, respectively) for the changes in slope.

The resulting distribution of profits and losses is shown in Fig (5). Note how the scenario produces very substantial losses and gains, and how each profit or loss can be associated with a configuration of upstream variables, and can therefore be interrogated. It is easy, but not discussed for the sake of brevity, to carry out sensitivity analysis.

12.2 Portfolio Applications
We call portfolio-based all those applications where the number of assets shocked is greater than the number of factors. In this case an arbitrary configuration of shocks to the asset prices will in general not be compatible with and any of the possible configurations for the driving factors, and the factor approach becomes indispensable to avoid the possibility of arbitrage (given the posited factor structure).

The highest hurdle for portfolio applications (especially for a portfolio management applications) is probably the recovery with a small number of factors of the normal-state expected returns and correlations for the representative assets of a realistically complex multi-currency portfolio. To test this aspect, we consider the case of a multi-currency portfolio, made up of dollar-, euro- and sterling-denominated assets (USD, EUR and GBP assets, respectively, in what follows). Fig (6) shows the expected returns obtained for a number of representative USD, EUR and GBP assets, using three factors (Expected Growth, Liquidity and Inflation) per currency. The column headed ‘Stat Ret’ presents the average of long-term statistical estimates for these

---

Figure 5: The profit and loss distribution associated with the duration and slope trades described in the text.
quantities obtained from a variety of econometric sources. The columns headed ‘LowER’ and ‘HiER’ show typical ranges for the representative assets from the various literature sources. The column headed ‘PredRet’ shows the expected returns produced by the two-stage regression with the factor model, as described in Section 9.1.

While the agreement is not perfect, the simultaneous agreement across currencies and asset classes between statistical and model-produced expected returns is very good. (The optimal risk-less rate turned out to be 0.0125.)

Fig (7) then shows the model-produced correlations among the assets in the three currencies. For the sake of brevity we do not report the statistical correlations among the different asset classes (just the topic of the equity/Treasury correlation has spawned in itself a huge literature). We simply draw the attention of the reader to the fact that many stylised empirical findings about correlation are recovered, such as: a negative correlation between safe-haven and risky assets; a positive correlation across currencies for assets of the same type (e.g., Treasury yields or equity indices); a very high correlation among different securities in the same currency and the same asset class; the high degree of correlations among USD, GBP and EUR Treasuries and among equity indices in different currencies; the low or negative correlation among risky assets (such as Emerging Market bonds) and safe-heaven assets (such as Treasuries); the very strong and positive correlation between assets in the same currency asset class (such as the 2-y and 10-y Treasury yields in USD, GBP and EUR); the monotonically decreasing correlation between Treasuries and credit yields of lower and lower quality. Among the less-positive features we note the correlation between equities and Treasuries (which is probably too negative), and among risky credit yields (among which the correlation is probably too high).

We stress that the example provided is simply a stylised proof-of-concept example, and that a considerable better quantitative agreement can be obtained, if one so wished, by adjusting, for instance, the possibly-correlated idiosyncratic factors, as explained in Section 9.1.

Figures (8), (9) and (10) then show: i) the profit and loss distribution in the normal state given a complex set of position in a USD, GBP and EUR portfolio; ii)

Figure 6: The expected returns obtained for a number of representative USD, EUR and GBP assets, using three factors (Expected Growth, Liquidity and Inflation) per currency. The column headed ‘Stat Ret’ presents the average of long-term statistical estimates for these quantities obtained from a variety of econometric sources. The columns headed ‘LowER’ and ‘HiER’ show typical ranges for the representative assets. The column headed ‘PredRet’ shows the expected returns produced by the two-stage regression with the factor model Section 9.1.

<table>
<thead>
<tr>
<th></th>
<th>vol</th>
<th>LowER</th>
<th>StatRet</th>
<th>HiER</th>
<th>PredRet</th>
</tr>
</thead>
<tbody>
<tr>
<td>2yTreas</td>
<td>2.0</td>
<td>0.50</td>
<td>0.75</td>
<td>1.00</td>
<td>1.19</td>
</tr>
<tr>
<td>10yTeas</td>
<td>6.0</td>
<td>2.00</td>
<td>2.25</td>
<td>2.50</td>
<td>2.29</td>
</tr>
<tr>
<td>2yBund</td>
<td>2.0</td>
<td>0.50</td>
<td>0.83</td>
<td>1.15</td>
<td>1.27</td>
</tr>
<tr>
<td>10yBund</td>
<td>4.5</td>
<td>0.75</td>
<td>1.13</td>
<td>1.50</td>
<td>1.46</td>
</tr>
<tr>
<td>2yGilt</td>
<td>6.0</td>
<td>0.50</td>
<td>0.75</td>
<td>1.00</td>
<td>0.32</td>
</tr>
<tr>
<td>10yGilt</td>
<td>6.0</td>
<td>2.50</td>
<td>3.25</td>
<td>4.00</td>
<td>2.58</td>
</tr>
<tr>
<td>USAAA</td>
<td>4.0</td>
<td>1.75</td>
<td>2.38</td>
<td>3.00</td>
<td>2.26</td>
</tr>
<tr>
<td>USAA</td>
<td>5.0</td>
<td>2.50</td>
<td>2.75</td>
<td>3.00</td>
<td>2.67</td>
</tr>
<tr>
<td>USA</td>
<td>7.0</td>
<td>3.00</td>
<td>3.25</td>
<td>3.50</td>
<td>3.51</td>
</tr>
<tr>
<td>USBBB</td>
<td>8.0</td>
<td>3.50</td>
<td>3.75</td>
<td>4.00</td>
<td>3.09</td>
</tr>
<tr>
<td>USHY</td>
<td>16.0</td>
<td>4.00</td>
<td>4.50</td>
<td>5.00</td>
<td>4.07</td>
</tr>
<tr>
<td>EM</td>
<td>25.0</td>
<td>6.00</td>
<td>8.00</td>
<td>10.00</td>
<td>8.63</td>
</tr>
<tr>
<td>S&amp;P</td>
<td>15.0</td>
<td>6.00</td>
<td>7.00</td>
<td>8.00</td>
<td>7.16</td>
</tr>
<tr>
<td>DAX</td>
<td>20.0</td>
<td>4.00</td>
<td>5.00</td>
<td>6.00</td>
<td>5.52</td>
</tr>
<tr>
<td>CAC</td>
<td>20.0</td>
<td>4.00</td>
<td>5.00</td>
<td>6.00</td>
<td>4.52</td>
</tr>
<tr>
<td>FTSE</td>
<td>20.0</td>
<td>5.00</td>
<td>6.00</td>
<td>7.00</td>
<td>5.59</td>
</tr>
</tbody>
</table>
the profit and loss distribution in the scenario state given the same complex set of position in the USD, GBP and EUR portfolio used to produce Fig (8); iii) the distribution of profits and losses associated with the 2-year and 1-year positions in the portfolio (with the changes in the 1-year rate obtained by the propagation technique described in Section 11). Explaining the portfolio sensitivities and the scenario would take too long a detour, but the figures provided give an idea of what the approach can offer.

Figure 7: The model-produced correlations among the assets in Fig (6).

Figure 8: The profit and loss distribution in the normal state given a complex set of position in a USD, GBP and EUR portfolio. Figures on the x axis in $ million.

Figure 9: The profit and loss distribution in the scenario state given the same complex set of position in the USD, GBP and EUR portfolio used to produce Fig (8). Figures on the x axis in $ million.
message conveyed by figures (8) and (9) is how significantly the ‘distressed’ profit and loss distribution can change from the normal state, towards which the distribution continuously converges as the event probability goes to zero.
13. Simple Extensions and Generalisations
The approach presented above attempts to provide a solid theoretical framework to implement a coherent and practically implementable stress-testing programme, but, clearly, many aspects have been left unexplored. However, a few simple extensions can be very easily accommodated. We therefore turn to these in this section.

First, we consider the case when stress testing is used at a transactional level (i.e., to analyse a specific trade), especially when the transaction is of the relative-value type. For prudential reasons, for instance, a risk manager may want to stress the vulnerability of a particular relative-value strategy (such as, for instance, the very common 10-year-future-versus-cash-bond trade). The procedure described above, based as it is on cascading shocks from high to granular level via a progressive coarsening of the ‘resolution’, will almost certainly fail to shine a spotlight on this specific corner. This can be fixed, however, by ‘promoting’ the spread of interest (the cash-versus-future spread) to the rank of a representative market index. After running the appropriate regressions (see Equation (10)) based on historical returns of the relative-value trade in question, everything then runs as above.

Second, we have spoken about shocking the factor-mimicking portfolios by a certain number of standard deviations, but we have not specified their marginal distributions. As a starting point, these can be approximated as Gaussian distributions (even in this case, the resulting shocks to the representative market indices will be far from normal, of course). For a given probability, this could understate the severity of a shock if a given factor-mimicking portfolio displayed very non-normal features.

Handling this only requires a trivial complication. One can start with the (possibly highly skewed and fat-tailed) empirical cumulative distribution, transform its random variables using the Probability Integral Transformation Theorem to random variables from the uniform distribution, and using the same theorem in reverse, transform back from the universe to the Gaussian \( \mathcal{N}(0, 1) \) distribution. The now-Gaussian marginals can then by conjoined by a Gaussian copula, and the easily computable moves of the jointly-normal associated variables can be obtained.

These shocks to the equivalent Gaussian variables are then translated back to the original distribution following the same path in reverse (from Gaussian to uniform draws, and from the uniform to the empirical marginal using the empirical inverse). If the Gaussian copula assumption is acceptable (or, at least, accepted) the procedure is very well known and totally standard. See, eg, Robert and Casella (2004)\(^{19}\) for a clear description of the procedure.

Finally, if one uses Bayesian nets, feedback loops, which can be important in finance in periods of market distress (see, eg, Cont and Schaaning, 2017), cannot be accommodated. There are well-established graph-theoretical techniques to handle the problem (see, eg, Denev, 2015). We neglect these extension to keep the already-long presentation as simple and focussed as possible.

19 - Sections 2.1, 2.2
14. Conclusions
We have presented a systematic approach to carrying out a stress testing programme that is rooted in solid asset-pricing theory; that satisfies several important desiderata (such as the consistency and continuity requirement), and which can be implemented in a practical and robust manner.

The presentation we have offered makes use of an underlying Bayesian net construction. We believe that the Bayesian-net technology is particularly well suited to the task, but we have shown how the approach can be extended to other stress-testing methodologies.

We have clearly outlined some limitations and shortcomings of the approach. As for the empirical validation of the approach, a substantial amount of empirical work remains to be carried out:

- the empirical effectiveness of the 'view' approach to a variety of combinations of representative/granular variables, which we have more asserted than proved, must be systematically investigated;
- the effectiveness of the 'risk spanning' given a set of chosen factor mimicking portfolios in a stress-testing context must also be empirically investigated; and
- the difficulty in creating self-financing portfolios that truly mimic a given macro factor should not be underestimated.

Each of these pieces of analysis would warrant a paper in its own right, and we therefore leave all of this for future work.

For all these shortcomings and the empirical work still to be done, we believe that the approach presented can offer a useful improvement on the current status of stress-testing practice and theory, and that it can make its practical application more widespread.
• Amenc N, Le Sourd V, (2003), Portfolio Theory and Performance Analysis, John Wiley, Chichester
• Corbae D, D’Erasmo P, Galaasen S, Irrarazabal A, Siemsen T, 2017, Structural Stress Tests,
• Dupont S, (2017), Accurate Valuation of Mid-Curve Options: Comparison with a LIBOR Market Model, Master’s Thesis, EDHEC Business School
• Glasserman P, Kang C, Kang W, 2014, Stress Scenario Selection by Empirical Likelihood, Quantitative Finance,
• International Monetary Fund, (2010), Macrofinancial Stress Testing — Principles and Practices, document prepared by the Monetary and Capital Markets Department, August 22, 2012
• Gaissendrees F, (2017), A New Method to Value Mid-Curve Options, Oxford University, Master’s Thesis in Mathematical Finance
• Parlatore C, Philippon T, 2018, Designing Stress Scenarios,
• Robert C P , Casella G, (2004), Monte Carlo Statistical Methods, Springer Verlag, Heidelberg and Berlin
About EDHEC-Risk Institute
About EDHEC-Risk Institute

Founded in 1906, EDHEC is one of the foremost international business schools. Operating from campuses in Lille, Nice, Paris, London and Singapore, EDHEC is one of the top 15 European business schools. Accredited by the three main international academic organisations, EQUIS, AACSB, and Association of MBAs, EDHEC has for a number of years been pursuing a strategy of international excellence that led it to set up EDHEC-Risk Institute in 2001. This Institute boasts a team of permanent professors, engineers and support staff, and counts a large number of affiliate professors and research associates from the financial industry among its ranks.

The Need for Investment Solutions and Risk Management

Investment management is justified as an industry only to the extent that it can demonstrate a capacity to add value through the design of dedicated and meaningful investor-centric investment solutions, as opposed to one-size-fits-all manager-centric investment products. After several decades of relative inertia, the much needed move towards investment solutions has been greatly facilitated by a true industrial revolution triggered by profound paradigm changes in terms of (1) mass production of cost- and risk-efficient smart factor indices; (2) mass customisation of liability-driven investing and goal-based investing strategies; and (3) mass distribution, with robo-advisor technologies. In parallel, the investment industry is strongly impacted by two other major external revolutions, namely the digital revolution and the environmental revolution.

In this fast-moving environment, EDHEC-Risk Institute positions itself as the leading academic think-tank in the area of investment solutions, which gives true significance to the investment management practice. Through our multi-faceted programme of research, outreach, education and industry partnership initiatives, our ambition is to support industry players, both asset owners and asset managers, in their efforts to transition towards a novel, welfare-improving, investment management paradigm.

EDHEC-Risk New Initiatives

In addition to the EDHEC Alternative Indexes, which are used as performance benchmarks for risk analysis by investors in hedge funds, and the EDHEC-IEIF Monthly Commercial Property index, which tracks the performance of the French commercial property market through SCPIs, EDHEC-Risk has recently launched a series of new initiatives.

- The EDHEC-Princeton Retirement Goal-Based Investing Index Series, launched in May 2018, which represent asset allocation benchmarks for innovative mass-customised target-date solutions for individuals preparing for retirement;

- The EDHEC Bond Risk Premium Monitor, the purpose of which is to offer to investment and academic communities a tool to quantify and analyse the risk premium associated with Government bonds;

- The EDHEC-Risk Investment Solutions (Serious) Game, which is meant to facilitate engagement with graduate students or investment professionals enrolled on one of EDHEC-Risk’s various campus-based, blended or fully-digital educational programmes.
About EDHEC-Risk Institute

Academic Excellence and Industry Relevance
In an attempt to ensure that the research it carries out is truly applicable, EDHEC has implemented a dual validation system for the work of EDHEC-Risk. All research work must be part of a research programme, the relevance and goals of which have been validated from both an academic and a business viewpoint by the Institute’s advisory board. This board is made up of internationally recognised researchers, the Institute’s business partners, and representatives of major international institutional investors. Management of the research programmes respects a rigorous validation process, which guarantees the scientific quality and the operational usefulness of the programmes.

Seven research programmes have been conducted by the centre to date:
• Investment Solutions in Institutional and Individual Money Management;
• Equity Risk Premia in Investment Solutions;
• Fixed-Income Risk Premia in Investment Solutions;
• Alternative Risk Premia in Investment Solutions;
• Multi-Asset Multi-Factor Investment Solutions;
• Reporting and Regulation for Investment Solutions;
• Technology, Big Data and Artificial Intelligence for Investment Solutions.

EDHEC-Risk Institute’s seven research programmes explore interrelated aspects of investment solutions to advance the frontiers of knowledge and foster industry innovation. They receive the support of a large number of financial companies. The results of the research programmes are disseminated through the EDHEC-Risk locations in the City of London (United Kingdom) and Nice, (France).

EDHEC-Risk has developed a close partnership with a small number of sponsors within the framework of research chairs or major research projects:
• Financial Risk Management as a Source of Performance, in partnership with the French Asset Management Association (Assocation Française de la Gestion financière – AFG);
• ETF, Indexing and Smart Beta Investment Strategies, in partnership with Amundi;
• Regulation and Institutional Investment, in partnership with AXA Investment Managers;
• Optimising Bond Portfolios, in partnership with BDF Gestion;
• Asset–Liability Management and Institutional Investment Management, in partnership with BNP Paribas Investment Partners;
• New Frontiers in Risk Assessment and Performance Reporting, in partnership with CACEIS;
• Exploring the Commodity Futures Risk Premium: Implications for Asset Allocation and Regulation, in partnership with CME Group;
• Asset–Liability Management Techniques for Sovereign Wealth Fund Management, in partnership with Deutsche Bank;
About EDHEC-Risk Institute

• The Benefits of Volatility Derivatives in Equity Portfolio Management, in partnership with Eurex;
• Innovations and Regulations in Investment Banking, in partnership with the French Banking Federation (FBF);
• Dynamic Allocation Models and New Forms of Target-Date Funds for Private and Institutional Clients, in partnership with La Française AM;
• Risk Allocation Solutions, in partnership with Lyxor Asset Management;
• Infrastructure Equity Investment Management and Benchmarking, in partnership with Meridiam and Campbell Lutyens;
• Risk Allocation Framework for Goal-Driven Investing Strategies, in partnership with Merrill Lynch Wealth Management;
• Financial Engineering and Global Alternative Portfolios for Institutional Investors, in partnership with Morgan Stanley Investment Management;
• Investment and Governance Characteristics of Infrastructure Debt Investments, in partnership with Natixis;
• Advanced Investment Solutions for Liability Hedging for Inflation Risk, in partnership with Ontario Teachers' Pension Plan;
• Cross-Sectional and Time-Series Estimates of Risk Premia in Bond Markets, in partnership with PIMCO;
• Active Allocation to Smart Factor Indices, in partnership with Rothschild & Cie;
• Solvency II, in partnership with Russell Investments;
• Advanced Modelling for Alternative Investments, in partnership with Société Générale Prime Services (Newedge);
• Structured Equity Investment Strategies for Long-Term Asian Investors, in partnership with Société Générale Corporate & Investment Banking.

The philosophy of the Institute is to validate its work by publication in international academic journals, as well as to make it available to the sector through its position papers, published studies and global conferences.

To ensure the distribution of its research to the industry, EDHEC-Risk also provides professionals with access to its website, https://risk.edhec.edu, which is devoted to international risk and investment management research for the industry. The website is aimed at professionals who wish to benefit from EDHEC-Risk’s analysis and expertise in the area of investment solutions. Its quarterly newsletter is distributed to more than 150,000 readers.
Research for Business
EDHEC-Risk Institute also has highly significant executive education activities for professionals, in partnership with prestigious academic partners. EDHEC-Risk’s executive education programmes help investment professionals upgrade their skills with advanced asset allocation and risk management training across traditional and alternative classes.

In 2012, EDHEC-Risk Institute signed two strategic partnership agreements. The first was with the Operations Research and Financial Engineering department of Princeton University to set up a joint research programme in the area of investment solutions for institutions and individuals. The second was with Yale School of Management to set up joint certified executive training courses in North America and Europe in the area of risk and investment management.

As part of its policy of transferring know-how to the industry, in 2013 EDHEC-Risk Institute also set up ERI Scientific Beta, which is an original initiative that aims to favour the adoption of the latest advances in smart beta design and implementation by the whole investment industry. Its academic origin provides the foundation for its strategy: offer, in the best economic conditions possible, the smart beta solutions that are most proven scientifically with full transparency in both the methods and the associated risks.

EDHEC-Risk Institute also contributed to the 2016 launch of EDHEC Infrastructure Institute (EDHECinfra), a spin-off dedicated to benchmarking private infrastructure investments. EDHECinfra was created to address the profound knowledge gap faced by infrastructure investors by collecting and standardising private investment and cash flow data and running state-of-the-art asset pricing and risk models to create the performance benchmarks that are needed for asset allocation, prudential regulation and the design of infrastructure investment solutions.
EDHEC-Risk Institute

2019

2018
• Goltz, F. and V. Le Sourd. The EDHEC European ETF and Smart Beta and Factor Investing Survey 2018 (August).
• Mantilla-Garcia, D. Maximising the Volatility Return: A Risk-Based Strategy for Homogeneous Groups of Assets (June).
• Martellini, L. and V. Milhau. Smart Beta and Beyond: Maximising the Benefits of Factor Investing (February).

2017
• Amenc, N., F. Goltz, V. Le Sourd. EDHEC Survey on Equity Factor Investing (November).
• Amenc, N., F. Goltz, V. Le Sourd. The EDHEC European ETF and Smart Beta Survey 2016 (May).
• Maeso, J.M., Martellini, L. Maximising an Equity Portfolio Excess Growth Rate: A New Form of Smart Beta Strategy? (November).
• Esakia, M., F. Goltz, S. Sivasubramanian and J. Ulahel. Smart Beta Replication Costs (February).
• Maeso, J.M., Martellini, L. Measuring Volatility Pumping Benefits in Equity Markets (February).
For more information, please contact:
Maud Gauchon on +33 (0)4 93 18 78 87
or by e-mail to: maud.gauchon@edhec-risk.com

EDHEC-Risk Institute
393 promenade des Anglais
BP 3116 - 06202 Nice Cedex 3
France
Tel. +33 (0)4 93 18 78 87

EDHEC Risk Institute—Europe
10 Fleet Place, Ludgate
London EC4M 7RB
United Kingdom
Tel: + 44 207 332 5600

risk.edhec.edu